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# Higher order loop equations for $A_r$ and $D_r$ quiver matrix models

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Abstract: We use free boson techniques to investigate $A$-$D$-$E$-quiver matrix models. Certain higher spin fields in the free boson formulation give rise to higher order loop equations valid at finite $N$. These fields form a special kind of $\mathcal{W}$-algebra, called Casimir algebra. We compute explicitly the loop equations for $A_r$ and $D_r$ quiver models and check that at large $N$ they are related to a deformation of the corresponding singular Calabi-Yau geometry.

Keywords: Conformal and W Symmetry, Matrix Models.




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## Contents



## 1. Introduction

Since the work of Kontsevich [1] it is well appreciated that matrix model techniques are very effective in solving topological string theory questions. Recently Dijkgraaf and Vafa proposed a new matrix model description for the topological string on non-compact Calabi-Yau threefolds, which are ALE fibrations of $A$-$D$-$E$ type over the complex plane [2]. Various checks have been made on the claim that the matrix model solves the closed topological string on these backgrounds [3, 4] and gives the exact holomorphic information of $\mathcal{N}=1$ quiver gauge theories, namely the superpotential and the gauge kinetic terms [5]. Orbifold and orientifold constructions widen the range of geometries and the field theories questions that can be addressed, see e.g. [6]–[9] and references therein.

The loop equations encode the symmetries of the matrix model as Ward identities, which are in principal strong enough to determine the correlators. Moreover the geometry of the local Calabi-Yau manifold is directly described by the large $N$ limit of the loop equations for the resolvent, which encodes the spectral density of the matrix.

One of the most elegant methods to obtain these equations is to rewrite the matrix model correlators in terms of CFT correlators in a theory of free chiral bosons. Correlators of products of bosonic currents are identified with matrix model correlators involving polynomials of the resolvent. Particular combinations of normal ordered products of the bosonic currents can be systematically identified with so-called Casimir fields. Their correlators fulfill a vanishing condition, which translates in the desired loop equations. This method is well known [10]–[12] and it was used in the context of the matrix model for the topological B-model to establish those terms in the loop equations of the $A$-quiver matrix models, which lead to the identification with the $A$ singularities in the fibre of the local Calabi-Yau space [2]. The Casimir fields form a $\mathcal{W}$-algebra, which is the symmetry algebra of the matrix model [13, 14]. As opposed to a normal Lie algebra, the commutator of two modes in a $\mathcal{W}$-algebra does not necessarily give only a finite sum of modes, but can also result in an infinite sum of products of modes. The special $\mathcal{W}$-algebra formed by the Casimir fields has been called a Casimir algebra [15, 16].

In the present paper we want to formulate the approach precisely enough to derive explicitly the exact finite $N$ loop equations for $A_r$ and $D_r$ quiver theories, which are our main result and are given in equations (3.14) and (3.43), (3.44). The techniques of section 2 equally apply to the $E$-series, but we do not work out loop equations explicitly for this case. The large $N$ loop equations enable us to identify the $A_r$ and $D_r$ fibre geometries together with the subleading terms, which encode the resolution of the singularity by renormalisable deformations, i.e. complex structure moduli of the topological string B-model.

These data calculate the exact $\mathcal{N}$=1 gauge theory information. The finite $N$ loop equations encode the exact terms of the gauge theory coupled to gravity. In the topological string context they encode the $g > 0$ genus correlators, which was checked for $A_2$ using the formalism of [17] in [3, 4] for genus 1. The equivalence between the generalised Konishi anomalies of the $\mathcal{N}$=1 theory and the loop equations, observed in the large $N$-limit in [18], was generalised to the gravitational sector in [19, 20].

The occurrence of the chiral CFT in the derivation of the loop equation seems auxiliary at the first glance. However it is known that the Virasoro constraints, which naturally appear in the CFT formulation, characterise topological string correlation functions for topological gravity, see [21] for review. The higher spin fields used to obtain the higher loop equations should lead to similar constraints for the topological string theory correlators in the quiver backgrounds. Recent results [22] for the open topological string established a chiral boson on a Riemann surface, which encodes the essential part of the Calabi-Yau geometry, as actually describing the complex structure deformation of the topological B-model. The Ward identities for the chiral boson correlators are strong enough to solve the all genus open string amplitudes for the topological vertex. Open/closed string duality extends this logic to the closed topological string. One should find a CFT formulation, which is identified with Kodaira-Spencer gravity for the closed string, and use the $\mathcal{W}$-constraints to directly fix the all genus partition function of the topological string on the quiver geometries.

Another motivation to study quiver matrix models comes from [23]. There it is argued that $\tau$-functions should be thought of as "the next generation of special functions", whose properties should be investigated on their own right and which will be relevant to describe partition functions of string theory. This program was started in [23] with the $\tau$-function given by the hermitean one matrix model, and among the obvious next candidates would be the quiver matrix models.

## 2. A-D-E quiver matrix models and free bosons

### 2.1 Loop equations and matrix models

Let us summarise the method we will use in the derivation of the quiver loop equations in the well-studied case of the hermitean one-matrix model, see e.g. the review [32]. The model is defined by the matrix integral

$$Z_{\text{mm}} = (\text{const}) \int d\Phi \, e^{-\frac{1}{g_s}\operatorname{Tr} W(\Phi)} = \int_{-\infty}^{\infty} d\lambda_1 \cdots d\lambda_N \prod_{\substack{m,n=1\\ m<n}}^{N} (\lambda_m - \lambda_n)^2 \, e^{-\frac{1}{g_s}\sum_{k=1}^{N} W(\lambda_k)} . \quad (2.1)$$

The first integral is over all hermitean $N \times N$ matrices, while the second integral amounts to expressing the first one in terms of the eigenvalues of $\Phi$. In the notation used below, this is the $A_1$-quiver model with the real axis as integration contour. The observables of the matrix model are traces of powers of $\Phi$,

$$Z_{\text{mm}}\left[\operatorname{Tr}(\Phi^{m_1}) \cdots \operatorname{Tr}(\Phi^{m_n})\right] = (\text{const}) \int d\Phi \operatorname{Tr}(\Phi^{m_1}) \cdots \operatorname{Tr}(\Phi^{m_n}) \, e^{-\frac{1}{g_s}\operatorname{Tr} W(\Phi)} . \quad (2.2)$$

These correlators can be conveniently expressed in terms of a generating function

$$Z_{\text{mm}}\left[\omega(z_1) \cdots \omega(z_n)\right] , \quad (2.3)$$

where we introduced the resolvent $\omega(z)$ of the matrix model, which is defined as

$$\omega(z) = \frac{1}{N} \operatorname{Tr} \frac{1}{z - \Phi} = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{z - \lambda_k} . \quad (2.4)$$

Expanding (2.3) in powers of $z_k$ gives as coefficients the correlators (2.2). The resolvent obeys the so-called loop equation, obtained by a change of variables $\Phi \to \Phi + \varepsilon/(z - \Phi)$ in the integral (2.1),

$$Z_{\text{mm}}\left[\omega(z)^2 - \frac{1}{N g_s} W'(z)\omega(z) + p(z)\right] = 0 , \quad (2.5)$$

where $p(z)$ is a polynomial of maximal degree equal to twice the degree of $W'(z)$. The loop equation determines the expectation value of the resolvent in the large $N$ limit, and it can also be used to set up an iterative procedure to find its $1/N$-expansion [24, 17].



JHEP03(2004)033

There is an alternative way to derive the loop equation (2.5) by relating the matrix model to the conformal field theory of one free boson in two dimensions via $Z_{\rm cft} = \frac{1}{N!} Z_{\rm mm}$ [10]–[12]. Here $Z_{\rm cft}$ is a shorthand for the vacuum correlator of the free boson in a complicated background which encodes the matrix integral. This relation is further used to map CFT correlators of $n$-fold normal ordered products of the free boson current $J(z)$ to matrix model correlators of polynomials in the resolvent,

$$Z_{\rm cft}\,[\,{:}J(z)\cdots J(z){:}] = ({\rm const}')Z_{\rm mm}\left[\left(\omega(z) - \frac{1}{2Ng_s}W'(z)\right)^n\right]. \qquad (2.6)$$

To recover the loop equations one needs in addition the relation

$$\oint_{\mathcal{C}} \frac{dz}{2\pi i}\,\frac{1}{z-x}\; Z_{\rm cft}\,[T(z)] = 0\,, \qquad (2.7)$$

where $T(z) = \frac{1}{2}{:}J(z)J(z){:}$ is the holomorphic stress tensor of the free boson and $\mathcal{C}$ is an integration contour specified in section 2.6. Also in this section, it is explained how (2.6) and (2.7) can be derived from the explicit form of $Z_{\rm cft}$. From (2.7) one finds that $Z_{\rm cft}[T(z)]$ is regular on all of $\mathbb{C}$, and in fact one can show that it is equal to a polynomial $\tilde{p}(z)$ of degree less or equal to twice the degree of $W'(z)$. Combining with (2.6) one finds

$$\tilde{p}(z) = \frac{1}{2} Z_{\rm cft}\,[{:}J(z)J(z){:}] = ({\rm const}'')\,Z_{\rm mm}\left[\left(\omega(z) - \frac{1}{2Ng_s}W'(z)\right)^2\right], \qquad (2.8)$$

which is the same as the loop equation (2.5) after expressing $\tilde{p}(z)$ in terms of $p(z)$.

For the quiver matrix models, this procedure is generalised in several ways. The quiver models are multi-matrix models (section 2.2) and accordingly we have not only one, but several resolvents $\omega_i(z)$. Further, these resolvents obey not only a loop equation of second order in the $\omega_i(z)$, but there is also a set of higher order loop equations. Attempting to find these equations by change of variables in the matrix integrals turns out to be extremely tedious, and exploiting the relation to CFT proves a far more efficient method. The CFT now consists of several free bosons (section 2.3), and not only the stress tensor leads to a loop equation, but there are also currents of spin higher than two, which obey a relation analogous to (2.7) and give corresponding higher order loop equations (sections 2.5 and 2.6).

These loop equations are not only essential for solving the matrix model, but describe also the background geometry of the type IIB string, whose low energy limit is the $\mathcal{N}{=}1$ ADE quiver gauge theory [2], see section 3.3 and 3.5 for the identification of the corresponding geometries from the large $N$ loop equations.

### 2.2 The A-D-E quiver matrix model

In general, a quiver matrix integral of rank $r$ takes the form [25, 26, 2]

$$Z_{\rm mm} = ({\rm const})\int \prod_{i=1}^{r} d\Phi_i \prod_{\langle m,n\rangle} dQ_{mn}\, e^{-\frac{1}{g_s}{\rm Tr}\,W(\Phi,Q)}\,. \qquad (2.9)$$

Here the $\Phi_i$ are $N_i \times N_i$ matrices and the $Q_{mn}$ are $N_m \times N_n$ matrices. The potential $W(\Phi, Q)$ is given by

$$W(\Phi, Q) = \sum_{i,j=1}^{r} s_{ij}\, Q_{ij}\, \Phi_j\, Q_{ji} + \sum_{i=1}^{r} W_i(\Phi_i)\,, \tag{2.10}$$

for some polynomials $W_i(x)$. The constants $s_{ij}$ are antisymmetric, $s_{ij} = -s_{ji}$, they obey $s_{ij} = 1$ if $i < j$ and the nodes in the quiver diagram are linked, and $s_{ij} = 0$ otherwise. The notation $\langle m, n\rangle$ for the range of the product in (2.9) denotes all pairs $(m, n)$ with $1 \leq m, n \leq r$ s.t. $s_{mn} \neq 0$. The definition of the integration region requires some care, see also [27]. We will address this problem in section 2.4.

In principle one can proceed by integrating out the $Q_{mn}$ and expressing $Z_{\mathrm{mm}}$ as an integral over the eigenvalues $\lambda_{i,I}$ of $\Phi_i$, where $I = 1, \ldots, N_i$. This would result in the expression

$$Z_{\mathrm{mm}} = \int \prod_{k,K} d\lambda_{k,K} \prod_{i=1}^{r} \prod_{\substack{I,J=1\\ I<J}}^{N_i} (\lambda_{i,I} - \lambda_{i,J})^2 \prod_{\substack{i,j=1\\ i<j}}^{r} \prod_{I=1}^{N_i} \prod_{J=1}^{N_j} (\lambda_{i,I} - \lambda_{j,J})^{-|s_{ij}|} e^{-\mathsf{S}}\,, \tag{2.11}$$

where

$$\mathsf{S} = \frac{1}{g_s} \sum_{l,L} W_l(\lambda_{l,L})\,, \tag{2.12}$$

and the constant in (2.9) has to be chosen appropriately. In practice this would require a consistent definition of the integration regions for the $\Phi_i$ and $Q_{mn}$ to avoid divergences. For the purpose of this paper we will take (2.11) — or rather a regularised form thereof, described in section 2.4 — as a definition of the matrix model and think of (2.9) as a motivation for considering an integral of the form (2.11).

From the general quiver models one obtains the A-D-E matrix models by choosing the $|s_{ij}|$ to take the special form

$$|s_{ij}| = 2\delta_{ij} - A_{ij}\,, \tag{2.13}$$

where $A_{ij}$ is the Cartan matrix of a rank $r$ Lie algebra of A-D-E type. In this case we have $A_{ij} = (\alpha_i, \alpha_j)$, where the $\alpha_i$ are the simple roots. The integral (2.11) can now be written more compactly as [25, 10, 11, 26, 2]

$$Z_{\mathrm{mm}} = \int \prod_{k,K} d\lambda_{k,K} \underbrace{\prod_{i,j=1}^{r} \prod_{I=1}^{N_i} \prod_{J=1}^{N_j}}_{(i,I)<(j,J)} (\lambda_{i,I} - \lambda_{j,J})^{(\alpha_i, \alpha_j)} e^{-\mathsf{S}}\,, \tag{2.14}$$

where one has to choose an ordering of the pairs $(i, I)$. Here we have chosen $(i, I) < (j, J)$ if either $i < j$ or else if $i = j$ and $I < J$. Different choices of ordering change equation (2.14) by a factor of $\pm 1$.

## 2.3 A free boson representation

In this section we will construct the integral (2.14) as a correlator in the chiral CFT of $r$ free bosons [10]–[12]. To fix conventions, consider one free boson $\varphi(z)$ for a start.

The U(1)-current $i\partial\varphi(z)$ will be denoted by $J(z)$. The vertex operators corresponding to the chiral part of the exponentials $:e^{iq\varphi(z)}:$ of the free boson field are called $V_q(z)$. They have U(1)-charge $q$ and conformal weight $h = \frac{1}{2}q^2$. The OPEs involving $J(z)$ read

$$J(z)J(w) = \frac{1}{(z-w)^2} + \mathrm{reg}(z-w)\,, \qquad J(z)V_q(w) = \frac{q}{z-w}V_q(w) + \mathrm{reg}(z-w) \tag{2.15}$$

and the conformal block with $n$ insertions on the complex plane, together with a charge $q$ placed at infinity, is given by

$$\langle q|V_{q_1}(z_1)\cdots V_{q_n}(z_n)|0\rangle = \delta_{q_1+\cdots+q_n,q} \prod_{i,j=1\,;\,i<j}^{n} (z_i - z_j)^{q_iq_j}\,. \tag{2.16}$$

Here $\langle q|$ is an out-state dual to the highest weight state $|q\rangle$ of U(1)-charge $q$. By definition we have

$$\langle q|V_{q_1}(z_1)\cdots V_{q_n}(z_n)|0\rangle = \lim_{L\to\infty} L^{q^2}\ \langle 0|V_{-q}(L)V_{q_1}(z_1)\cdots V_{q_n}(z_n)|0\rangle\,. \tag{2.17}$$

The modes of the current $J(z)$ are defined via

$$J(z) = \sum_{k\in\mathbb{Z}} J_k\, z^{-k-1} \quad , \quad J_k = \oint_{\gamma_0} \frac{dz}{2\pi i}\, z^k\, J(z)\,, \tag{2.18}$$

where $\gamma_0$ is a circular contour around the origin. The OPEs (2.15) imply the commutation relations

$$[J_m, J_n] = m\,\delta_{m+n,0}\,, \qquad [J_m, V_q(z)] = qz^m\, V_q(z)\,. \tag{2.19}$$

Using these it follows that for $k \geq 0$

$$J_k V_{q_1}(z_1)\cdots V_{q_n}(z_n)|0\rangle = (\sum_{i=1}^{n} q_i(z_i)^k)\ V_{q_1}(z_1)\cdots V_{q_n}(z_n)|0\rangle\,, \tag{2.20}$$

and thus also, for any out-state $\langle X|$ and $k \geq 0$,

$$\langle X|e^{-tJ_k}V_{q_1}(z_1)\cdots V_{q_n}(z_n)|0\rangle = e^{-t\sum_{i=1}^{n} q_i(z_i)^k}\langle X|V_{q_1}(z_1)\cdots V_{q_n}(z_n)|0\rangle\,. \tag{2.21}$$

Let us now return to the general case. Denote by $\mathsf{g}$ a simply laced Lie algebra of rank $r$. Let $K$ be the Killing form on $\mathsf{g}$ and fix once and for all a basis $H^a$, $a = 1,\ldots,r$ of the Cartan subalgebra $\mathsf{h}$ of $\mathsf{g}$ s.t. $K(H^a, H^b) = \delta_{a,b}$. Let $\mathsf{h}^*$ be the dual of $\mathsf{h}$ and denote by $e_a$ the basis dual to $H^a$. Another basis of $\mathsf{h}^*$ is provided by the $r$ simple roots $\alpha_i$ of $\mathsf{g}$. The bilinear form on $\mathsf{h}^*$ induced by $K$ is denoted by $(\cdot,\cdot)$.

Consider the chiral CFT consisting of the product of $r$ free bosons, with components $J^{(a)}(z)$ and $V_q^{(a)}(z)$, where $a = 1,\ldots,r$. To an element $u = \sum_a u^a e_a$ of $\mathsf{h}^*$ assign the current $J^u(z)$ and the vertex operator $V_u(z)$ as

$$J^u(z) = \sum_{a=1}^{r} u_a\, J^{(a)}(z) \qquad \text{and} \qquad V_u(z) = \prod_{a=1}^{r} V_{u_a}^{(a)}(z)\,. \tag{2.22}$$

One computes the conformal weight of $V_u(z)$ to be $h(V_u) = \frac{1}{2}(u,u)$. With the conventions (2.22), the analogues of formulas (2.15), (2.16) and (2.19) are given by

$$\begin{aligned}
&J^u(z)J^v(w) = \frac{(u,v)}{(z-w)^2} + \mathrm{reg}(z-w)\,, \qquad J^u(z)V_q(w) = \frac{(u,q)}{z-w}V_q(w) + \mathrm{reg}(z-w)\\
&\langle q|V_{q_1}(z_1)\cdots V_{q_n}(z_n)|0\rangle = \delta_{q_1+\cdots+q_n,q}\prod_{i,j=1\,;i<j}^{n}(z_i-z_j)^{(q_i,q_j)}\,,\\
&[J^u_m, J^v_n] = (u,v)\,m\,\delta_{m+n,0}\,, \qquad [J^u_m, V_q(z)] = (u,q)\,z^m\,V_q(z)\,.
\end{aligned} \tag{2.23}$$

Here $u, v, q$ as well as $q_1, \dots, q_n$ are elements of $\mathsf{h}^*$. The modes of $J^u(z)$ are written as $J^u_k$.

To the $r$ potentials $W_i(x)$ in (2.10) we assign a $\mathsf{h}^*$-valued function $\mathsf{W}(x)$ as follows. Suppose

$$W_i(x) = \sum_{k=0}^{\infty} t_k^{(i)} x^k \qquad \text{for } i = 1, \dots, r \tag{2.24}$$

and define vectors $\tau_k \in \mathsf{h}^*$ via $(\tau_k, \alpha_i) = t_k^{(i)}$. We then set $\mathsf{W}(x) = \sum_{k\geq 0}\tau_k\, x^k \in \mathsf{h}^*$. This potential enters the definition of an operator $\mathsf{H}$, which encodes the potentials of the matrix model in the free boson representation,

$$\mathsf{H} = \frac{1}{g_s}\sum_{a=1}^{r}\oint_{\gamma_0}\frac{dz}{2\pi i}\,(\mathsf{W}(z), e_a)\; J^{(a)}(z) = \frac{1}{g_s}\sum_{k\geq 0} J_k^{\tau_k}\,. \tag{2.25}$$

As a final ingredient we need the screening charges

$$Q_i = \oint_{\gamma_i}\frac{dz}{2\pi i} V_{\alpha_i}(z)\,, \tag{2.26}$$

where $\gamma_i$ are some integration contours, to be specified in section 2.4. Note that because $\mathsf{g}$ is simply laced we have $(\alpha_i, \alpha_i) = 2$ for all $i$ and the vertex operators $V_{\alpha_i}(z)$ indeed have conformal weight one.

The free boson representation of the quiver matrix integral $Z_{\mathrm{mm}}$ in (2.14) is now given by [10, 11, 26, 12]

$$Z_{\mathrm{cft}} = \langle\vec{N}|e^{-\mathsf{H}}e^{Q_1}\cdots e^{Q_r}|0\rangle \qquad \text{where} \qquad \langle\vec{N}| = \langle N_1\alpha_1 + \cdots + N_r\alpha_r|\,, \tag{2.27}$$

which is equal to $Z_{\mathrm{mm}}$ up to a $N_i$-dependent constant

$$Z_{\mathrm{cft}} = C_{\vec{N}}\, Z_{\mathrm{mm}} \qquad \textit{where} \qquad C_{\vec{N}} = \frac{1}{N_1!\cdots N_r!}\,. \tag{2.28}$$

To see this equality note that

$$\begin{aligned}
Z_{\mathrm{cft}} &= \frac{1}{N_1!\cdots N_r!}\langle\vec{N}|\,e^{-\frac{1}{g_s}\sum_k J_k^{\tau_k}}\,(Q_1)^{N_1}\cdots(Q_r)^{N_r}\,|0\rangle\\
&= \frac{1}{N_1!\cdots N_r!}\left(\prod_{k,K}\oint_{\gamma_k}\frac{d\lambda_{k,K}}{2\pi i}\right)e^{-\frac{1}{g_s}\sum_\ell\sum_{i,I}(\tau_\ell,\alpha_i)(\lambda_{i,I})^\ell}\,\langle\vec{N}|\prod_{j,J}V_{\alpha_j}\left(\lambda_{j,J}\right)|0\rangle\,,
\end{aligned} \tag{2.29}$$

where in the first step we inserted the definition of $\mathsf{H}$ and used that by conservation of the $r$ U(1)-charges only one term of the exponentials of the screening charges in (2.27) can contribute. In the second step the screening charges have been replaced by the corresponding integrals and the relation (2.21) has been employed to remove the modes $J_k^{\tau_k}$. Inserting the explicit expression for the $r$-boson conformal blocks in (2.23) one recovers the integral (2.14).

### 2.4 Integration contours

As already mentioned, in this paper we will consider (2.9) as a formal expression motivating the eigenvalue representation (2.14), which we will take as a definition of the A-D-E quiver matrix model. However, even in (2.14) the integrand contains potential singularities, namely whenever $(\alpha_i, \alpha_j) < 0$ and two eigenvalues belonging to the nodes $i$ and $j$ approach each other. Below, in a procedure similar to [27], the expression (2.14) is defined as a limit of a regularised integral. The regularisation involves a number of ad hoc choices, but these turn out not to affect the loop equations we will derive.

Let $\gamma : \mathbb{R} \to \mathbb{C}$ be a smooth contour parametrised such that it has unit tangent $|\dot\gamma(t)| = 1$. Given a small $\varepsilon > 0$ define a family of contours $\gamma_{k,K}^{\varepsilon,L}$ on the interval $[L, -L]$ as

$$\gamma_{k,K}^{\varepsilon,L}(t) = \gamma(t) + \frac{i\,\varepsilon}{K + \sum_{j=1}^{k-1} N_j}\dot\gamma(t) \ , \tag{2.30}$$

see figure 1. With these contours we define a regularised integral $I^{\mathrm{reg}(\varepsilon,L)}[\,\cdot\,]$ as

$$I^{\mathrm{reg}(\varepsilon,L)}\left[f(\lambda_{1,1},\ldots)\right] = \left(\prod_{k=1}^{r}\prod_{K=1}^{N_k}\oint_{\gamma_{k,K}^{\varepsilon,L}}\frac{d\lambda_{k,K}}{2\pi i}\right) f(\lambda_{1,1},\ldots)\,. \tag{2.31}$$

Here it is implied that the contours $\gamma_{k,K}^{\varepsilon,L}$ are integrated on the interval $[L, -L]$ only.

Using the integral operator (2.31) one defines the regularised partition functions $Z_{\mathrm{mm}}^{\mathrm{reg}(\varepsilon,L)}$ and $Z_{\mathrm{cft}}^{\mathrm{reg}(\varepsilon,L)}$. With this choice of contour the various eigenvalues $\lambda_{k,K}$ are always at a finite distance from each other and $Z_{\mathrm{mm}}^{\mathrm{reg}(\varepsilon,L)} = Z_{\mathrm{cft}}^{\mathrm{reg}(\varepsilon,L)}/C_{\vec N}$ is a (finite) number, because it is defined as a multiple integral over a finite region of a bounded function.

Below we will be working with

$$Z_{\mathrm{mm}}^{\mathrm{reg}} = \lim_{L\to\infty} Z_{\mathrm{mm}}^{\mathrm{reg}(\varepsilon,L)} \tag{2.32}$$

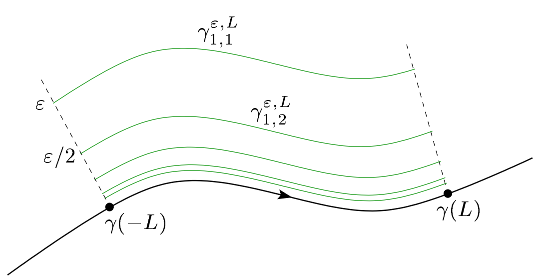


**Figure 1:** The original integration contour $\gamma$ and the shifted and truncated contours $\gamma_{k,K}^{\varepsilon,L}$ used in the regularisation prescription.

which a priory still depends on $\varepsilon$. However the loop equations turn out to be independent of this parameter. Note also that requiring the limit (2.32) to be finite imposes further constraints on the initial contour $\gamma$. The regularised integral (2.14) converges if (but not only if) $\mathrm{Re}(W_i(\gamma(L))) \to \infty$ as $L \to \pm\infty$ for all $1 \le i \le r$.

### 2.5 Higher spin currents commuting with screening charges

The loop equations in the matrix model will be constructed from fields in the chiral algebra, which commute with the screening charges. These fields will be called *Casimir fields* for a reason explained in section 4.

Before turning to the definition of Casimir fields we recall some notations to express the OPE of chiral fields. We will use the conventions of [28, section 6.5]. For two chiral fields $A(z)$, $B(z)$ the chiral fields $\{AB\}_n(z)$ are defined via

$$A(z)B(w) = \sum_{n=-\infty}^{n_0} \frac{\{AB\}_n(w)}{(z-w)^n} . \tag{2.33}$$

The (generalised) normal ordered product $(AB)(z)$ of two chiral fields is defined to be

$$(AB)(z) = \oint_{\gamma_z} \frac{dx}{2\pi i} \frac{A(x)B(z)}{x-z} = \{AB\}_0(z) , \tag{2.34}$$

where $\gamma_z$ is a circular contour winding tightly around $z$. Multiple normal ordered products are defined recursively, e.g. $(ABCD)(z) \equiv (A(B(CD)))(z)$. In general this form of normal ordering is neither associative nor commutative, see e.g. [28, appendix 6.C], or the [15, appendix] for details. We will also be using this generalised normal ordering for free boson fields, where it reduces to the usual notion of moving creation operators to the left.

Consider a general spin $s$ chiral field $W(z)$,

$$W(z) = \sum_{n=1}^{s} \sum_{a_1,\ldots,a_n} \sum_{\substack{m_1,\ldots,m_n \ge 0 \\ m_1+\cdots+m_n = s-n}} d_{a_1\ldots a_n}^{m_1\ldots m_n} \left( \partial^{m_1} J^{(a_1)} \cdots \partial^{m_n} J^{(a_n)} \right)(z) . \tag{2.35}$$

We will give two equivalent definitions for $W(z)$ to be a Casimir field.

**Definition 1.** *The spin $s$ chiral field $W(z)$ is a Casimir field if $V_0 W(0)|0\rangle = 0$, for all $V(z) = V_{\alpha_i}(z)$, $i = 1, \ldots, r$. Here $V_0$ denotes the zero mode of the current $V(z)$.*

**Definition 2.** *$W(z)$ is a Casimir field if, for $i = 1, \ldots, r$, the OPE of $V_{\alpha_i}(x)$ with $W(z)$ is a total derivative in $x$,*

$$V_{\alpha_i}(x)W(z) = \frac{\partial}{\partial x}\left( \sum_n (x-z)^n A_n(z) \right) , \tag{2.36}$$

*where the $A_n(z)$ are some chiral fields.*

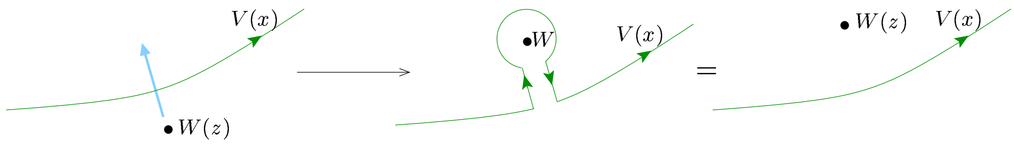


**Figure 2:** If a Casimir field $W(z)$ is analytically continued through a screening charge integral, the screening integral can be deformed back to its original form. Casimir fields commute with screening integrals.

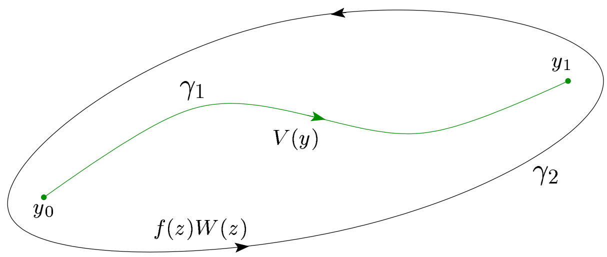


**Figure 3:** Integral of a Casimir field $W(z)$ times a function $f(z)$ (contour $\gamma_2$) around a regularised screening charge integral with contour $\gamma_1 = \gamma_{k,K}^{\varepsilon,K}$.

To see that the two definitions are equivalent, first note that, by the state-field correspondence, $V_0 W(0)|0\rangle = 0$ is equivalent to $\oint_{\gamma_z} \frac{dx}{2\pi i} V(x)W(z) = \{VW\}_1(z) = 0$. Inserting (2.36) in the last expression, we see that definition 2 implies definition 1. Conversely, if $\{VW\}_1(z) = 0$ we can write

$$V(x)W(z) = \sum_{n=-\infty}^{n_0} \frac{\{VW\}_n(z)}{(x-z)^n} = \frac{\partial}{\partial x}\left( \sum_{\substack{n=-\infty \\ n\neq 1}}^{n_0} \frac{\{VW\}_n(z)}{(1-n)(x-z)^{n-1}} \right) \tag{2.37}$$

so that definition 1 implies definition 2.

As we have seen while arguing the equality of the two definitions, a Casimir field $W(z)$ has the property $\oint_{\gamma_z} \frac{dx}{2\pi i} V(x)W(z) = 0$, i.e. a contour integral of $V(x)$ around $W(z)$ can be left out. It follows that $W(z)$ does not have a discontinuity across the screening charge integration contours, as illustrated in figure 2. This is the first important property of Casimir fields. The second property appears when integrating $W(z)$ around a regularised screening charge contour $\gamma_{k,K}^{\varepsilon,L}$ as in figure 3, where $\gamma_1 = \gamma_{k,K}^{\varepsilon,L}$ and $y_0 = \gamma_1(-L)$, $y_1 = \gamma_1(L)$. The function $f(z)$ is assumed to be analytic inside the $z$-integration contour $\gamma_2$, and will later on be set to $f(z) = (z-x)^{-1}$ for some point $x$ outside the contour $\gamma_2$. The double integration indicated in figure 3 can be solved

$$\oint_{\gamma_1} \frac{dy}{2\pi i} \oint_{\gamma_2} \frac{dz}{2\pi i} f(z) \cdot W(z)V(y) = A(y_1) - A(y_0)\,, \tag{2.38}$$

with $A(y)$ given by

$$A(y) = \sum_{k=0}^{n_0-2} \sum_{l=0}^{n_0-k-2} \frac{(-1)^{k+l}\ \partial^l f(y)}{k!\, l!\, (k+l+1)}\ \partial^k \{VW\}_{k+l+2}(y)\,. \tag{2.39}$$

In verifying this calculation one can substitute the OPE (2.37), then Taylor-expand both $\{VW\}_n(z)$ and $f(z)$ around the point $y$ and carry out the $z$-integration, which gives rise to a Kronecker delta, removing the $n$-summation. The $y$-integration can be solved trivially because of the derivative $\partial/\partial y$ introduced by the OPE.

The ability (2.38) of $W(z)$ to replace screening charge integrations by field insertions at the integration boundaries is crucial in the following argument, which ultimately leads to the loop equations.

### 2.6 From Casimir fields to loop equations

As a generalisation of (2.27) let us introduce the notation

$$Z_{\rm cft}^{\rm reg}[\phi_1(x_1)\cdots\phi_n(x_n)] = \langle \vec{N}| e^{-\mathsf{H}} e^{Q_1}\cdots e^{Q_r} \phi_1(x_1)\cdots\phi_n(x_n)|0\rangle \tag{2.40}$$

for the (unnormalised) expectation value of some chiral fields $\phi_k(z)$ in the matrix model background. Consider the equation

$$\lim_{L\to\infty} \oint_{\gamma_{\rm ev}} \frac{dz}{2\pi i}\ \frac{1}{z-x}\ Z_{\rm cft}^{{\rm reg}(\varepsilon,L)}[W(z)] = 0\,, \tag{2.41}$$

where the contour $\gamma_{\rm ev}$ surrounds all contours $\gamma_{k,K}^{\varepsilon,L}$, but not the point $x$. This equation was given in [13, 14, 10] in terms of modes, and in integral form (with the $L$-limit implicit) in [26, 12] (for the stress tensor) as well as in [2]. Equation (2.41) will be established in detail in appendix A. The outline of the argument is as follows. The integration contour $\gamma_{\rm ev}$ is deformed to encircle each one of the regularised contours $\gamma_{k,K}^{\varepsilon,L}$. By (2.38) such an integral can be replaced by an insertion of $A(y_1) - A(y_0)$. In the limit $L\to\infty$ this insertion will cause an exponential damping in the correlator, due to the non-trivial out-state $\langle\vec{N}|e^{-\mathsf{H}}$.

The rewriting of (2.41) as a loop equation will proceed in several steps. First note the commutation relations

$$[J_k^u, \mathsf{H}^n] = \frac{1}{g_s} k\,\theta(-k)\ (u,\tau_{-k})\ n\,\mathsf{H}^{n-1}\,, \tag{2.42}$$

which can be verified by recursion. Here $\theta(x)$ is the Heaviside function. As a short calculation shows, this in turn implies that

$$e^{-\mathsf{H}} J^u(z) = \left( J^u(z) - \frac{1}{g_s}\left(u, \mathsf{W}'(z)\right)\right) e^{-\mathsf{H}} \qquad \text{with} \qquad \mathsf{W}'(z) = \frac{\partial}{\partial z}\mathsf{W}(z)\,. \tag{2.43}$$

Define the positive and negative mode part of $J^u(z)$ as

$$J_+^u(z) = \sum_{k\geq 0} J_k^u\, z^{-k-1} \qquad \text{and} \qquad J_-^u(z) = \sum_{k<0} J_k^u\, z^{-k-1} \tag{2.44}$$

so that $J^u(z) = J^u_+(z) + J^u_-(z)$. Since $\langle\vec{N}|J^u_k = 0$ for all $k < 0$ it is easy to see that $\langle\vec{N}|J^u(z) = \langle\vec{N}|J^u_+(z)$. In fact, using the definition of the normal ordered product one can verify the relation

$$\langle\vec{N}|(\partial^{m_1}J^{u_1}\cdots\partial^{m_n}J^{u_n})(z) = \langle\vec{N}|\partial^{m_1}J^{u_1}_+(z)\cdots\partial^{m_n}J^{u_n}_+(z)\,. \tag{2.45}$$

Putting all this together we arrive at the conclusion

$$\begin{aligned} &Z^{\rm reg}_{\rm cft}\left[(\partial^{m_1}J^{u_1}\cdots\partial^{m_1}J^{u_n})(z)\right] = \\ &\quad = \langle\vec{N}|\prod_{k=1}^{n}\left(\partial^{m_k}J^{u_k}_+(z) - \frac{1}{g_s}\left(u_k, \partial^{m_k+1}\mathsf{W}(z)\right)\right)e^{-\mathsf{H}}e^{Q_1}\cdots e^{Q_r}|0\rangle\,. \end{aligned} \tag{2.46}$$

On the matrix model side, we have $r$ different resolvents $\omega_k(z)$, defined as

$$\omega_k(z) = \frac{1}{N}\sum_{K=1}^{N_k}\frac{1}{z-\lambda_{k,K}} \qquad \text{where} \quad N = N_1 + \cdots + N_r\,. \tag{2.47}$$

Expectation values of the resolvents are defined as

$$\begin{aligned} &Z^{\rm reg}_{\rm mm}\left[\omega_{a_1}(z_1)\ldots\omega_{a_s}(z_s)\right] = \\ &\quad = I^{\rm reg}\left[\omega_{a_1}(z_1)\ldots\omega_{a_s}(z_s)\underbrace{\prod_{i,j=1}^{r}\prod_{I=1}^{N_i}\prod_{J=1}^{N_j}}_{(i,I)<(j,J)}(\lambda_{i,I}-\lambda_{j,J})^{(\alpha_i,\alpha_j)}\; e^{-\frac{1}{g_s}\sum_{l,L}W_l(\lambda_{l,L})}\right]. \end{aligned} \tag{2.48}$$

The link between (2.46) and (2.48) is provided by the equations

$$\left(-g_s\sum_{k\geq 0}z^{-k-1}\frac{\partial}{\partial\tau^{(a)}_k}\right)e^{-\mathsf{H}} = J^{(a)}_+(z)e^{-\mathsf{H}} \tag{2.49}$$

and

$$\left(-\frac{g_s}{N}\sum_{k\geq 0}z^{-k-1}\frac{\partial}{\partial t^{(i)}_k}\right)e^{-\mathsf{S}} = \omega_i(z)e^{-\mathsf{S}}\,. \tag{2.50}$$

Here we have set $\tau^{(a)}_k = (\tau_k, e_a)$. Recall that by definition $t^{(i)}_k = (\tau_k, \alpha_i) = \sum_{a=1}^{r}(\alpha_i, e_a)\tau^{(a)}_k$, which results in a corresponding relation for the partial derivatives, namely

$$\frac{\partial}{\partial\tau^{(a)}_k} = \sum_{i=1}^{r}(e_a,\alpha_i)\,\frac{\partial}{\partial t^{(i)}_k}\,. \tag{2.51}$$

In this way one obtains, for example,

$$\begin{aligned} Z^{{\rm reg}(\varepsilon,L)}_{\rm cft}\left[J^{(a)}(z)\right] &= \left(-g_s\sum_{k\geq 0}z^{-k-1}\frac{\partial}{\partial\tau^{(a)}_k} - \frac{1}{g_s}\left(e_a, \mathsf{W}'(z)\right)\right)Z^{{\rm reg}(\varepsilon,L)}_{\rm cft} \\ &= \left(-N\sum_{i=1}^{r}(e_a,\alpha_i)\frac{g_s}{N}\sum_{k\geq 0}z^{-k-1}\frac{\partial}{\partial t^{(i)}_k} - \frac{1}{g_s}\left(e_a, \mathsf{W}'(z)\right)\right)\;C_{\vec{N}}\;Z^{{\rm reg}(\varepsilon,L)}_{\rm mm} \\ &= C_{\vec{N}}\;N\;Z^{{\rm reg}(\varepsilon,L)}_{\rm mm}\left[\sum_{i=1}^{r}(e_a,\alpha_i)\omega_i(z) - \frac{1}{Ng_s}\left(e_a, \mathsf{W}'(z)\right)\right]. \end{aligned} \tag{2.52}$$

Note, however, that in writing (2.50) we have expanded $(z-\lambda_{k,K})^{-1} = z^{-1}\sum_{j>0}(\lambda_{k,K}/z)^j$. This is possible only if $|z| > |\lambda_{k,K}|$. For finite $L$ we can always choose a large enough $R_0(L)$ such that all contours $\gamma_{k,K}^{\varepsilon,L}$ are contained in a disc of radius $R_0(L)$ centered at the origin. From (2.49) and (2.50) we thus know that (2.52) holds for all $z$ outside of this disc, i.e. $|z| > R_0(L)$. For $|z| \leq R_0(L)$ the relation then holds by analytic continuation in $z$. Since we have established (2.52) for all values of $L$ and $z$, it will continue to hold if we take the limit $L \to \infty$.

In general we set, for $u \in \mathsf{h}^*$,

$$y_u(z) = \sum_{i=1}^{r} (u, \alpha_i)\, \omega_i(z) - \frac{1}{N g_s} \big(u, \mathsf{W}'(z)\big)\,. \tag{2.53}$$

In the $L \to \infty$ limit we then obtain the following relation between the CFT expectation value of a normal ordered product of free boson currents and the matrix model expectation value of a polynomial in the resolvents:

$$Z_{\rm cft}^{\rm reg}\left[(\partial^{m_1} J^{u_1} \cdots \partial^{m_n} J^{u_n})(z)\right] = C_{\vec N}\, N^n\ Z_{\rm mm}^{\rm reg}\left[\partial^{m_1} y_{u_1}(z) \cdots \partial^{m_n} y_{u_n}(z)\right]. \tag{2.54}$$

This equation, together with (2.35) and (2.41) gives the desired relation between a spin $s$ Casimir field $W(z)$ and a loop equation of maximal order $s$ in the resolvents. Concretely, suppose we are given a Casimir field in the form (2.35). Then

$$\oint_{\gamma_{\rm ev}} \frac{dz}{2\pi i}\, \frac{1}{z-x}\ Z_{\rm cft}^{{\rm reg}(\varepsilon,L)}[W(z)] = C_{\vec N}\, N^s \left(F(x) + P(x)\right), \tag{2.55}$$

where

$$\begin{aligned} F(x) &= -\sum_{n=1}^{s} N^{n-s} \sum_{a_1,\dots,a_s} \sum_{\substack{m_1,\dots,m_n\geq 0 \\ m_1+\dots+m_n = s-n}} d_{a_1\dots a_n}^{m_1\dots m_n} \times \\ &\qquad \times Z_{\rm mm}^{{\rm reg}(\varepsilon,L)}\Big[\, \partial^{m_1} y_{(a_1)}(x) \cdots \partial^{m_n} y_{(a_n)}(x)\Big] \qquad (2.56)\\ P(x) &= \sum_{n=1}^{s} N^{n-s} \sum_{a_1,\dots,a_s} \sum_{\substack{m_1,\dots,m_n\geq 0 \\ m_1+\dots+m_n = s-n}} d_{a_1\dots a_n}^{m_1\dots m_n} \times \\ &\qquad \times \oint_{\gamma_\infty} \frac{dz}{2\pi i}\, \frac{1}{z-x} Z_{\rm mm}^{{\rm reg}(\varepsilon,L)}\left[\partial^{m_1} y_{(a_1)}(z) \cdots \partial^{m_n} y_{(a_n)}(z)\right] \qquad (2.57) \end{aligned}$$

as can be seen by deforming the contour $\gamma_{\rm ev}$ into a sum of two contours, one encircling the point $x$ and the other, denoted by $\gamma_\infty$, a large circle containing all the eigenvalues as well as the point $x$. Here we also abbreviated $y_{(a)}(z) = y_{e_a}(z)$.

The function $P(x)$ is analytic in the entire complex plane. Furthermore the lhs of (2.55) behaves as $c/x$ for some constant $c$ as $x$ tends to infinity. On the other hand, since for the resolvents we have $\omega_i(x) \cong N_i/N \cdot x^{-1}$ as $x \to \infty$ and since the potential terms with higher derivatives are subleading in $x$, the function $F(x)$ behaves as

$$F(x) = \frac{(-1)^{s+1}}{(N g_s)^s}\ Z_{\rm mm}^{{\rm reg}(\varepsilon,L)} \sum_{a_1,\dots,a_s} d_{a_1\dots a_s}^{0\dots 0} \left(e_{a_1}, \mathsf{W}'(x)\right) \cdots \left(e_{a_s}, \mathsf{W}'(x)\right) + O(x^{sD-1}), \tag{2.58}$$

where

$$D = \max\left(\deg(W_i'(x)) \mid i = 1, \ldots, r\right). \tag{2.59}$$

In order for $F(x) + P(x)$ to behave as $c/x$ for $x \to \infty$, $P(x)$ thus has to be a polynomial of maximal degree $sD$. Taking the $L \to \infty$ limit in (2.55), one finally arrives at the loop equation associated to a Casimir field in the form (2.35),

$$Z_{\rm mm}^{\rm reg}\left[\sum_{n=1}^{s} N^{n-s} \sum_{a_1,\ldots,a_s} \sum_{\substack{m_1,\ldots,m_n \geq 0 \\ m_1+\cdots+m_n=s-n}} d_{a_1\ldots a_n}^{m_1\ldots m_n}\ \partial^{m_1} y_{(a_1)}(x) \cdots \partial^{m_n} y_{(a_n)}(x) + P_s(x)\right] = 0\,, \tag{2.60}$$

where $P_s(x) = -P(x)/Z_{\rm mm}$ is a polynomial of degree $\deg(P_s) \leq sD$.

### 2.7 Undetermined parameters in the loop equations

In order to use the loop equations to determine the matrix model correlators recursively, we need to know more about the a priori undetermined polynomials $P_s(x)$ appearing in (2.60). Let us for this section assume that all of the $W_i(x)$ have degree $D{+}1$. Writing out the product of $y$'s in the integrand of $P(x)$ in (2.57), one finds the leading terms to be of the form $(\mathsf{W}'(z))^s + \omega(z)(\mathsf{W}'(z))^{s-1} + N^{-1}\mathsf{W}''(z)(\mathsf{W}'(z))^{s-2} + \ldots$, where we have omitted all sums, indices, etc. Because of the asymptotics of $\omega(z)$ one sees that the (a priori unknown) resolvents start to enter $P(x)$ only at order $x^{D(s-1)-1}$, while the coefficients of $x^{Ds}$ to $x^{D(s-1)}$ are directly determined through the potential $\mathsf{W}(x)$. It follows that the polynomial $P_s(x)$ entering a loop equation of order $s$ has $D(s{-}1)$ undetermined coefficients.

On general grounds one expects an independent loop equation of order $s$ for every Lie algebra Casimir of $\mathsf{g}$ of order $s$. This will be motivated in section 4, where the relation between Lie algebra Casimirs and a special kind of $\mathcal{W}$-algebra, called Casimir algebras, is discussed.

The Lie algebra $\mathsf{g}$ has $r$ Casimir operators, whose order $s$ is related to the exponents $e$ of $\mathsf{g}$ via $s = e{+}1$. The exponents obey the relation $\sum_{e\in\exp(\mathsf{g})} e = R_+$, where $R_+ = \frac{1}{2}(\dim\mathsf{g}{-}r)$ denotes the number of positive roots of $\mathsf{g}$. The overall number of undetermined coefficients in the polynomials $P_s(x)$ of the $r$ independent loop equations now reads

$$\#(\text{undet. coeff}) = \sum_{e\in\exp(\mathsf{g})} D(e{+}1{-}1) = DR_+\,. \tag{2.61}$$

The same number of free parameters appears when setting up a large $N$-expansion. The eigenvalues of the matrices $\Phi_i$ are located on the cuts of the resolvents $\omega_i(z)$. For small enough $g_s$ the cuts will be located around the solutions of the classical equations of motion, which correspond to the critical points of the quiver potential (2.10). There are $DR_+$ such critical points [29, 30, 2] and for the large $N$ expansion one has to fix the corresponding $DR_+$ filling fractions. One can thus expect a (however quite nontrivial) bijection between the filling fractions and the parameters (2.61).

It should be remarked that the large $N$ expansion is in general just a formal procedure and does not necessarily reproduce the value of the matrix integrals for partition function and correlators, see [31].

### 2.8 The quadratic loop equation as an example

In this section we recover the familiar feature that the quadratic loop equation is linked to the Virasoro algebra [13, 14], see also e.g. [11, 32].

Recall that the OPE of a weight $h$ primary field $\phi(z)$ with the conformal stress tensor $T(z)$ is of the form

$$T(z)\phi(w) = \frac{h}{(z-w)^2}\,\phi(w) + \frac{1}{z-w}\,\frac{\partial}{\partial w}\,\phi(w) + \mathrm{reg}(z-w)\,. \tag{2.62}$$

This can always be written as a total derivative in $w$ if the conformal weight $h$ is equal to one. Since all the screening charge currents $V_{\alpha_i}(z)$ are primary and have conformal weight one, $T(z)$ is a Casimir field. In terms of the U(1)-currents of the $r$ free bosons we have

$$T(z) = \sum_{a=1}^{r} \frac{1}{2} \left( J^{(a)} J^{(a)} \right) (z)\,. \tag{2.63}$$

Applying equation (2.60) to this Casimir field and substituting (2.53) yields the quadratic loop equation for the quiver models

$$Z_{\mathrm{mm}}^{\mathrm{reg}} \left[ \sum_{i,j=1}^{r} A_{ij}\ \omega_i(x)\,\omega_j(x) - \frac{2}{Ng_s} \sum_{i=1}^{r} W_i'(x)\,\omega_i(x) + P(x) \right] = 0\,. \tag{2.64}$$

Here $A_{ij} = (\alpha_i, \alpha_j)$ is the Cartan matrix of $\mathsf{g}$ and we used the identity $(\alpha_i, \mathsf{W}(x)) = W_i(x)$, which follows from the definition of $\mathsf{W}(x)$ below (2.24). The summand $(\mathsf{W}'(x), \mathsf{W}'(x))/(Ng_s)^2$ has been combined with $2P_2(x)$ to the polynomial $P(x)$, which has maximal degree $2D$, but at most $D$ coefficients which are not directly determined by $\mathsf{W}(x)$, see section 2.7. Quadratic loop equations have been found for similar multi-matrix models in [26, 12], and in [33] the large $N$ limit of (2.64) was given for the $A_r$-case.

## 3. Examples

### 3.1 $A_r$-quiver model loop equations at finite $N$

In this section we want to apply the techniques introduced above to find a set of $r$ loop equations for the $A_r$-quiver matrix model. We have

$$\text{exponents of} \quad A_r = \{1, 2, \ldots, r-1, r\} \tag{3.1}$$

and thus expect the loop equations to have orders $2, 3, \ldots, r, r+1$. We will start their construction by introducing a convenient set of $r+1$ vectors $\{\varepsilon_1, \ldots, \varepsilon_{r+1}\}$ in $\mathsf{h}^*$, corresponding to the weights of the vector representation of $A_r$,

$$\varepsilon_i = \sum_{k=i}^{r} \alpha_k - \sum_{k=1}^{r} \frac{k}{r+1}\,\alpha_k\,, \qquad \text{for} \qquad i = 1, \ldots, r+1\,, \tag{3.2}$$

where $\alpha_1, \ldots, \alpha_r$ are the $r$ simple roots of $A_r$. Using also $(\alpha_i, \alpha_j) = A_{ij} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$ one can verify the properties

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}\,, \qquad \sum_{k=1}^{r+1} \varepsilon_k = 0\,, \qquad (\varepsilon_i, \varepsilon_j) = \delta_{i,j} - \frac{1}{r+1}\,. \tag{3.3}$$

As a further ingredient we will need the commutation relations of the modes $J^u_k$ with the modes of the currents $V_{\alpha_k}(z)$, which we will denote by $V_{\alpha_k,m}$. These follow from (2.23) to be

$$[J^u_m, V_{\alpha_k,n}] = (u, \alpha_k) V_{\alpha_k,m+n}\,, \qquad \text{where} \qquad V_{\alpha_k,m} = \oint_{\gamma_0} \frac{dz}{2\pi i}\, z^m\, V_{\alpha_k}(z)\,. \tag{3.4}$$

In writing the contour integral for the modes $V_{\alpha_k,m}$ we used that $V_{\alpha_k}(z)$ has conformal weight one. Using (3.3) and (3.4), a short calculation shows that, for any $\mu \in \mathbb{C}$,

$$\left[V_{\alpha_k,0}, \left(\mu + J^{\varepsilon_k}_{-1}\right)\left(\mu + J^{\varepsilon_{k+1}}_{-1}\right)\right] = J^{\alpha_k}_{-1} V_{\alpha_k,-1} - V_{\alpha_k,-2}\,. \tag{3.5}$$

Proceeding analogously as in [34, 16] we define a state $|R_{r+1}\rangle$ (closely connected to a quantum Miura transformation),

$$|R_{r+1}\rangle = \prod_{i=1}^{r+1} (\mu + J^{\varepsilon_i}_{-1})|0\rangle\,, \tag{3.6}$$

which will serve as a generating function for $r$ Casimir fields.

In fact, consider a field $W(z)$ of the form (2.35), but without derivatives. Writing out the definition of the normal ordered products, one verifies

$$W(0)|0\rangle = \sum_{a_1,\ldots,a_s}^{r} d_{a_1\ldots a_s} J^{(a_1)}_{-1} \cdots J^{(a_s)}_{-1}|0\rangle\,. \tag{3.7}$$

In the same way, the field $R_{r+1}(z)$ corresponding to (3.6), i.e. the field with the property $R_{r+1}(0)|0\rangle = |R_{r+1}\rangle$ is just given by the normal ordered product

$$R_{r+1}(z) = (A^1 A^2 \ldots A^{r+1})(z) \qquad \text{where} \qquad A^k(z) = \mu \mathbb{1} + J^{\varepsilon_k}(z)\,. \tag{3.8}$$

We would like to show that, for any $\mu \in \mathbb{C}$, $R_{r+1}(z)$ is a Casimir field. To this end we verify definition 1 given in section 2.5. Consider the transformations

$$\begin{aligned} V_{\alpha_k,0}|R_{r+1}\rangle &= \prod_{i=1}^{k-1}(\mu + J^{\varepsilon_i}_{-1})\left[V_{\alpha_k,0}, (\mu + J^{\varepsilon_k}_{-1})(\mu + J^{\varepsilon_{k+1}}_{-1})\right] \prod_{i=k+2}^{r+1}(\mu + J^{\varepsilon_i}_{-1})|0\rangle \\ &= \prod_{i=1}^{k-1}(\mu + J^{\varepsilon_i}_{-1})(J^{\alpha_k}_{-1} V_{\alpha_k,-1} - V_{\alpha_k,-2}) \prod_{i=k+2}^{r+1}(\mu + J^{\varepsilon_i}_{-1})|0\rangle \\ &= \prod_{\substack{i=1 \\ i \notin \{k,k+1\}}}^{r+1} (\mu + J^{\varepsilon_i}_{-1})(J^{\alpha_k}_{-1} V_{\alpha_k,-1} - V_{\alpha_k,-2})|0\rangle\,. \end{aligned} \tag{3.9}$$

Here, in the first step we used that by (3.4) $V_{\alpha_k,0}$ has nontrivial commutation relations only with $J^{\varepsilon_k}_{-1}$ and $J^{\varepsilon_{k+1}}_{-1}$. In the second step (3.5) is substituted, and finally this part of the expression is commuted past the right product, using that $V_{\alpha_k,m}$ has trivial commutation relations with $J^{\varepsilon_i}_{-1}$ if $i \geq k{+}2$.

Next we will investigate the states $J^{\alpha_k}_{-1} V_{\alpha_k,-1}|0\rangle$ and $V_{\alpha_k,-2}|0\rangle$, and show that they are in fact equal. This will imply that (3.9) is equal to zero, and hence $R_{r+1}(z)$ is indeed a Casimir field.

First note that by definition

$$|\alpha_k\rangle = V_{\alpha_k}(0)|0\rangle = \oint_{\gamma_0} \frac{dz}{2\pi i}\,\frac{1}{z}\, V_{\alpha_k}(z)|0\rangle = V_{\alpha_k,-1}|0\rangle\,. \tag{3.10}$$

Further, the state $V_{\alpha_k,-2}|0\rangle$ has $L_0$-eigenvalue 2 and charge $\alpha_k$. However, all states with $L_0$-eigenvalue 2 and charge $\alpha_k$ are of the form $J^u_{-1}|\alpha_k\rangle$, for $u \in \mathsf{h}^*$. To find which is the correct value of $u$, we compose both states with $\langle\alpha_k|J^v_1$ from the left. Using (2.23) and (3.4) one easily checks

$$\langle\alpha_k|J^v_1 J^u_{-1}|\alpha_k\rangle = (v,u)\langle\alpha_k|\alpha_k\rangle \qquad \text{and} \qquad \langle\alpha_k|J^v_1 V_{\alpha_k,-2}|0\rangle = (v,\alpha_k)\langle\alpha_k|\alpha_k\rangle\,. \tag{3.11}$$

For this to hold true for any $v \in \mathsf{h}^*$ one needs to have $u = \alpha_k$. We have thus established

$$V_{\alpha_k,-2}|0\rangle = J^{\alpha_k}_{-1}|\alpha_k\rangle\,. \tag{3.12}$$

Knowing that $R_{r+1}(z)$ is a Casimir field for any $\mu \in \mathbb{C}$, we can expand in $\mu$ and find the individual Casimir fields $W^{(s)}(z)$ of spin $s = 1,\ldots,r{+}1$ as, see [34] and also e.g. [35, 10, 16],

$$W^{(s)}(z) = \sum_{1\leq i_1<\ldots<i_s\leq r+1} (J^{\varepsilon_{i_1}}\cdots J^{\varepsilon_{i_s}})(z)\,. \tag{3.13}$$

Note that $W^{(1)} = 0$ since the $\varepsilon_k$ sum to zero. We can now apply the relation (2.60) between Casimir fields and loop equations to all of the $W^{(s)}(z)$. The result can again be conveniently written in terms of a generating function

$$Z^{\text{reg}}_{\text{mm}} \left[ \prod_{k=1}^{r+1} (\mu{+}y_{\varepsilon_k}(x)) + \sum_{k=0}^{r+1} \mu^{r+1-k} P_k(x) \right] = 0\,, \tag{3.14}$$

where $P_k(x)$ is a polynomial of degree $\leq kD$. The functions $y_{\varepsilon_k}(x)$ are expressed in terms of the resolvents and the potential according to (2.53). Setting $\omega_0(x) = 0 = \omega_{r+1}(x)$ we can write

$$y_{\varepsilon_k}(x) = \omega_k(x) - \omega_{k-1}(x) + t_k(x)\,. \tag{3.15}$$

Using (3.2) together with $(\alpha_i, \mathsf{W}'(x)) = W'_i(x)$ we find that the tree level potentials of quiver model action (2.12) enter in the combination

$$t_k(x) = -\frac{1}{g_s N}(\varepsilon_k, \mathsf{W}'(x)) = \frac{1}{g_s N}\left( \sum_{i=1}^{r} \frac{i}{r{+}1} W'_i(x) - \sum_{i=k}^{r} W'_i(x) \right). \tag{3.16}$$

Note that (3.15) can easily be inverted,

$$\omega_i(x) = \sum_{k=1}^{i} \left(y_{\varepsilon_k}(x) - t_k(x)\right). \tag{3.17}$$

Expanding (3.14) in $\mu$ gives rise to $r$ loop equations for the $A_r$-quiver matrix model, which hold at finite $N$. The equations arising at the powers $\mu^{r+1}$ and $\mu^r$ are trivial.

### 3.2 A closer look at the cubic and quartic loop equations

The cubic loop equation of the $A_r$ model corresponds to the coefficient of $\mu^{r-2}$ in the $\mu$-expansion of (3.14). A priori it is given as an ordered sum over $y_{\varepsilon_i} y_{\varepsilon_j} y_{\varepsilon_k}$. The ordered sum can however be simplified by rewriting it in terms of sums over the full index range and using $\sum_{i=1}^{r+1} y_{\varepsilon_i} = 0$. Explicitly

$$\sum_{1\le i<j<k\le r+1} y_{\varepsilon_i} y_{\varepsilon_j} y_{\varepsilon_k} = \frac{1}{3!}\sum_{i,j,k=1}^{r+1} y_{\varepsilon_i} y_{\varepsilon_j} y_{\varepsilon_k} - \frac{1}{2}\sum_{i,k=1}^{r+1} (y_{\varepsilon_i})^2 y_{\varepsilon_k} + \frac{1}{3}\sum_{i=1}^{r+1} (y_{\varepsilon_i})^3 , \tag{3.18}$$

where on the rhs all but the last term sum to zero. This yields the following form for the cubic loop equation, valid for general $r$,

$$Z_{\rm mm}^{\rm reg}\left[\sum_{i=1}^{r+1} (y_{\varepsilon_i})^3 + 3P_3\right] = 0 . \tag{3.19}$$

Then, using $\omega_0 = 0 = \omega_{r+1}$, $\varepsilon_i = \alpha_i + \varepsilon_{i+1}$ and defining $\eta = N g_s$, one obtains

$$\begin{aligned} Z_{\rm mm}^{\rm reg}\Bigg[\sum_{i=1}^{r}\Bigg(&\omega_{i-1}\omega_i^2 - \omega_{i-1}^2\omega_i + \frac{1}{\eta}W_i'\left(\omega_i^2 - \frac{1}{\eta}\omega_i W_i'\right) + \\ &+\frac{2}{\eta}\left(\omega_i^2 - \omega_i\omega_{i+1} - \frac{1}{\eta}\omega_i W_i'\right)(\varepsilon_{i+1}, \mathsf{W}')\Bigg) + Q_3\Bigg] = 0 , \end{aligned} \tag{3.20}$$

for some polynomial $Q_3(x)$ of degree $\le 3D$. This form of the cubic can be transformed further by making use of the quadratic loop equation (2.64). Substituting $\varepsilon_{i+1} = \varepsilon_2 - \sum_{k=2}^{i} \alpha_k$ into (3.20), the coefficient of $(\varepsilon_2, \mathsf{W}')$ is just a polynomial (due to the quadratic loop equation) and can be absorbed into $Q_3$, resulting in

$$\begin{aligned} Z_{\rm mm}^{\rm reg}\Bigg[\sum_{i=1}^{r}&\left(\omega_{i-1}\omega_i^2 - \omega_{i-1}^2\omega_i + \frac{1}{\eta}W_i'\left(\omega_i^2 - \frac{1}{\eta}W_i'\omega_i\right)\right) - \\ &-\frac{2}{\eta}\sum_{i=2}^{r}\left(\omega_i^2 - \omega_i\omega_{i+1} - \frac{1}{\eta}W_i'\omega_i\right)\sum_{k=2}^{i} W_k' + \tilde{Q}_3\Bigg] = 0 , \end{aligned} \tag{3.21}$$

where $\tilde{Q}_3(x)$ is some polynomial of degree $\le 3D$. For $r = 2$ this equation reduces to the one in [7].

The quartic loop equation can be treated in the same way, taking the coefficient of $\mu^{r-3}$ in (3.14). For $A_3$ this coefficient is just $y_{\varepsilon_1} y_{\varepsilon_2} y_{\varepsilon_3} y_{\varepsilon_4}$ and the loop equation reads explicitly

$$\begin{aligned} Z_{\rm mm}^{\rm reg}[\,&(3W_1' + 2W_2' + W_3' - 4\eta\omega_1)(W_1' - 2W_2' - W_3' - 4\eta\omega_1 + 4\eta\omega_2)\times \\ &\times(W_1' + 2W_2' + 3W_3' - 4\eta\omega_3)(W_1' + 2W_2' - W_3' - 4\eta\omega_2 + 4\eta\omega_3) - 256\eta^4 P_4] = 0 . \end{aligned} \tag{3.22}$$

For general $r$ it is convenient to rewrite the ordered sum over four indices similarly as was done above for three indices, giving the $A_r$ quartic loop equation in the form

$$Z_{\rm mm}^{\rm reg}\Big[\sum_{i,k=1}^{r+1}(y_{\varepsilon_i})^2(y_{\varepsilon_k})^2-2\sum_{i=1}^{r+1}(y_{\varepsilon_i})^4+8P_4\Big]=0\,. \tag{3.23}$$

### 3.3 $A_r$-quiver model loop equations at large $N$

In the large $N$ limit (keeping $N_i/N$ and $g_sN$ fixed) the $n$-point functions in the matrix model factorise into products of $n$ one-point functions. The loop equations encoded in (3.14) become algebraic. Abbreviating $\tilde{y}_{\varepsilon_k}(x)=Z_{\rm mm}^{\rm reg}[y_{\varepsilon_k}(x)]/Z_{\rm mm}^{\rm reg}$ and replacing $\mu\to z$ we can write

$$\prod_{k=1}^{r+1}(\,z+\tilde{y}_{\varepsilon_k}(x)\,)=-\sum_{k=0}^{r+1}P_k(x)z^{r+1-k}\,, \tag{3.24}$$

where again $P_k(x)$ is a polynomial of degree $\leq kD$. It follows that, given the polynomials $P_k(x)$, the $\tilde{y}_{\varepsilon_k}(x)$ are minus the $r{+}1$ roots of the polynomial in $z$ on the rhs of (3.24), i.e. minus the $r{+}1$ solution to the equation

$$z^{r+1}-\sum_{k=0}^{r-1}P_{r+1-k}(x)z^k=0\,, \tag{3.25}$$

where we also used that (3.24) forces $P_0(x)=-1$ and $P_1(x)=0$. This also proves a claim in [2] that the loop equations of the $A_r$-quiver matrix models reproduce, in the large $N$ limit, a deformed reduction to one complex dimension of the singular, non-compact Calabi-Yau threefold geometry associated to the $A_r$-quiver. Explicitly, the non-compact Calabi-Yau three-fold is given by the fibration of an $A_r$-singularity over the complex $x$-plane, [36, 29, 2],

$$u^2+v^2+\prod_{k=1}^{r+1}(z+t_k(x))=0\,. \tag{3.26}$$

Here the $t_k(x)$ as given by (3.16) are polynomials, fixed by the tree level potentials, and hence the geometry reduced with respect to the $u$ and $v$ direction is a nodal curve. In the quantum geometry as described by the matrix model loop equation the $t_k(x)$ are replaced by $\tilde{y}_{\epsilon_k}(x)$ and the $DR_+$ parameters dicussed in section 2.7 are turned on. The curve does not factorise algebraically any more and the double points get resolved. Note that the number of parameters $DR_+$, which are not fixed at tree level is sufficient to resolve all double points.

Topological string amplitudes of the B-model and exact gauge theory quantities are calculated in terms of periods of the resolved curve with respect to the meromorphic differentials, which are reductions of the Calabi-Yau $(3,0)$ form over $r$ cycles of (3.26). They are given by

$$\eta_i=\tilde{y}_{\alpha_i}(x)\mathrm{d}x=\left(\sum_{j=1}^{r}A_{ij}\,\omega_j(z)-\frac{1}{Ng_s}W_i'(x)\right)\mathrm{d}x\,, \tag{3.27}$$

where we set $u=\alpha_i$ in (2.53) and used $(\alpha_i,\mathsf{W}'(x))=W_i'(x)$.

Note that expanding out (3.24) gives the $r$ loop equations as ordered sums,

$$\sum_{1\leq i_1<\cdots<i_s\leq r+1} \tilde{y}_{\varepsilon_{i_1}}(x)\ldots\tilde{y}_{\varepsilon_{i_s}}(x)+P_s(x)=0\,. \qquad (3.28)$$

A corresponding equation has been conjectured in [33] to arise from an analysis of the Konishi anomaly in quiver gauge theories. Using $\sum_{i=1}^{r+1}\tilde{y}_{\varepsilon_i}(x)=0$, equation (3.28) can be rewritten in the more concise form

$$\sum_{i=1}^{r+1}\tilde{y}_{\varepsilon_i}(x)^s+Q_s(x)=0\,, \qquad (3.29)$$

for some polynomial $Q_s(x)$ of degree $\leq sD$. The equivalence of (3.28) and (3.29) can be seen recursively. One first checks the statement for $s$=2. For general $s$ one rewrites the ordered sum in terms of several sums over the full index range and uses the already established identities (3.29) of degree less than $s$. For example in the case $s$=3 one rewrites the ordered sum as in (3.18), where only the last term survives. In the case $s$=4 the same procedure leads to the large $N$ form of (3.23); using the case $s$=2, one can replace $\sum_{i,j=1}^{r+1}\tilde{y}_{\varepsilon_i}(x)^2\tilde{y}_{\varepsilon_j}(x)^2=Q_2(x)^2$ to obtain the form (3.29).

### 3.4 $D_r$-quiver model loop equations at finite $N$

Next we turn to the investigation of $D$-series. For the Lie algebra $D_r$ with $r\geq 3$ we have

$$\text{exponents of}\quad D_r=\{\,r-1\,,\,1,3,\ldots,2r-3\,\}\,. \qquad (3.30)$$

Accordingly we expect a generating set of Casimir fields of spin $r$ and spins $2,4,\ldots,2r-2$. The procedure of [37], which we will recall below, is to start by constructing the Casimir field $W^{(r)}(z)$ of spin $r$ and then find the fields of spin $2,4,\ldots 2r-2$ as elements of the OPE $W^{(r)}(z)W^{(r)}(w)$.

As for $A_r$ we first need to choose a convenient set of vectors $\varepsilon_i$ in $\mathsf{h}^*$. In the case of $D_r$ we choose $r$ such vectors, which form a basis of $\mathsf{h}^*$. Let the simple roots of $D_r$ be numbered such that the Cartan matrix $A_{ij}=(\alpha_i,\alpha_j)$ is of the form

$$A_{ij}=2\delta_{i,j}-\delta_{i,j+1}-\delta_{i,j-1}+\delta_{i,r}(\delta_{j,r-1}-\delta_{j,r-2})+\delta_{j,r}(\delta_{i,r-1}-\delta_{i,r-2})\,. \qquad (3.31)$$

The vectors $\varepsilon_i$, for $i=1,\ldots,r$ are defined as

$$\varepsilon_i=\sum_{k=i}^{r-2}\alpha_k\;+\;\frac{1}{2}\Big(\alpha_r+(-1)^{\delta_{i,r}}\alpha_{r-1}\Big)\,, \qquad (3.32)$$

where for negative range the sum is taken to be zero. The vectors $\varepsilon_1,\ldots,\varepsilon_r$ have the properties

$$(\varepsilon_i,\varepsilon_j)=\delta_{i,j}\,,\qquad \alpha_i=\varepsilon_i-(1-\delta_{i,r})\varepsilon_{i+1}+\delta_{i,r}\varepsilon_{r-1}\,. \qquad (3.33)$$

Using the relations (3.4) and (3.12) (which are not specific to $A_r$ but valid for any simply laced Lie algebra) one can check the equations, for $i\leq r-1$,

$$\begin{aligned} V_{\alpha_i,0}J_{-1}^{\varepsilon_i}J_{-1}^{\varepsilon_{i+1}}|0\rangle &= J_{-1}^{\alpha_i}V_{\alpha_i,-1}|0\rangle-V_{\alpha_i,-2}|0\rangle=0\,,\\ V_{\alpha_r,0}J_{-1}^{\varepsilon_{r-1}}J_{-1}^{\varepsilon_r}|0\rangle &= -J_{-1}^{\alpha_r}V_{\alpha_r,-1}|0\rangle+V_{\alpha_r,-2}|0\rangle=0\,. \end{aligned} \qquad (3.34)$$

Define the field $W^{(r)}(z)$ via

$$W^{(r)}(0)|0\rangle = J^{\varepsilon_1}_{-1} J^{\varepsilon_2}_{-1} \cdots J^{\varepsilon_r}_{-1}|0\rangle\,. \tag{3.35}$$

Since all of the $r$ $J$-modes entering (3.35) commute and also each of the screening charge zero modes $V_{\alpha_i,0}$ commutes with all but two of the $J^{\varepsilon_k}$ one finds

$$\begin{aligned} V_{\alpha_i,0} W^{(r)}(0)|0\rangle &= \left( \prod_{\substack{k=1 \\ k\notin\{i,i+1\}}}^{r} J^{\varepsilon_k}_{-1} \right) V_{\alpha_i,0} J^{\varepsilon_i}_{-1} J^{\varepsilon_{i+1}}_{-1}|0\rangle = 0\,, \\ V_{\alpha_r,0} W^{(r)}(0)|0\rangle &= \left( \prod_{k=1}^{r-2} J^{\varepsilon_k}_{-1} \right) V_{\alpha_r,0} J^{\varepsilon_{r-1}}_{-1} J^{\varepsilon_r}_{-1}|0\rangle = 0\,. \end{aligned} \tag{3.36}$$

Thus $W^{(r)}(z)$ is indeed a Casimir field. Since by translation invariance also $[V_{\alpha_i,0}, W^{(r)}(z)] = 0$, we have in particular that, for $i = 1, \ldots, r$ and all values of $z$,

$$V_{\alpha_i,0} W^{(r)}(z) W^{(r)}(0)|0\rangle = 0\,. \tag{3.37}$$

It follows that upon writing out the OPE

$$W^{(r)}(z)\, W^{(r)}(0)\, |0\rangle = \sum_{m=-\infty}^{2r} z^{-m}\, \widetilde{W}^{(2r-m)}(0)|0\rangle\,, \tag{3.38}$$

every coefficient $\widetilde{W}^{(2r-m)}(0)|0\rangle$ is a state corresponding to a Casimir field. This gives an a priori infinite number of Casimir fields, but they will not all be independent. However, one can show that the $W^{(r)}(z)$ together with $\widetilde{W}^{(s)}(z)$, $s = 2, 4, \ldots, 2r-2$ do indeed form an independent set of Casimir fields, see [37] and e.g. [16, section 6.3.3] .

Let us investigate the fields $\widetilde{W}^{(s)}(z)$, $s = 2, 4, \ldots, 2r-2$ in more detail. In terms of the modes $W_m$ of $W^{(r)}(z)$, defined via $W^{(r)}(z) = \sum_m z^{-m-r} W_m$, the OPE (3.38) takes the form

$$W^{(r)}(z) W^{(r)}(0)|0\rangle = \sum_{m\in\mathbb{Z}} z^{-m-r} W_m W_{-r}|0\rangle\,, \tag{3.39}$$

so that $\widetilde{W}^{(s)}(0)|0\rangle = W_{r-s}W_{-r}|0\rangle$. Using the definition of the normal ordered product, it is not difficult to show that

$$\begin{aligned} W^{(r)}(z) = (J^{\varepsilon_1} \cdots J^{\varepsilon_r})(z) &= \sum_{m_1,\ldots,m_r\in\mathbb{Z}} z^{-(m_1+\cdots+m_r+r)}\, {:}J^{\varepsilon_1}_{m_1} \cdots J^{\varepsilon_r}_{m_r}{:}\,, \\ W_m &= \sum_{\substack{m_1,\ldots,m_r\in\mathbb{Z} \\ m_1+\cdots+m_r=m}} {:}J^{\varepsilon_1}_{m_1} \cdots J^{\varepsilon_r}_{m_r}{:}\,, \end{aligned} \tag{3.40}$$

where $:\cdots:$ denotes the usual normal ordering of modes, such that the positive modes are to the right. Because of (2.23) and (3.33) all the $J$-modes in (3.40) commute and the normal ordering can be omitted. If a term in the sum (3.40) for $W_m$ is to contribute to the



JHEP03(2004)033

product $W_m W_{-r}|0\rangle$, we need the $m_k$ to be in the set $S = \{1\} \cup \mathbb{Z}_{\leq -1}$. This gives explicitly

$$\begin{aligned}\widetilde{W}^{(s)}(0)|0\rangle &= \sum_{\substack{m_1,\dots,m_r\in S\\ m_1+\cdots+m_r=r-s}} J^{\varepsilon_1}_{m_1}J^{\varepsilon_1}_{-1}\cdots J^{\varepsilon_r}_{m_r}J^{\varepsilon_r}_{-1}|0\rangle\\ &= \sum_{n=1}^{s/2}\sum_{1\leq i_1<\ldots<i_n\leq r}\sum_{\substack{m_1,\dots,m_n\geq 0\\ m_1+\cdots+m_n=s-2n}} J^{\varepsilon_{i_1}}_{-m_1-1}J^{\varepsilon_{i_1}}_{-1}\cdots J^{\varepsilon_{i_n}}_{-m_n-1}J^{\varepsilon_{i_n}}_{-1}|0\rangle\,. \qquad (3.41)\end{aligned}$$

The second expression for $\widetilde{W}^{(s)}(0)|0\rangle$ is obtained form the first by cancelling all $J^{\varepsilon_k}_{m_k}J^{\varepsilon_k}_{-1}$ where $m_k = 1$ and redefining the other $m_k$ as $m_k \rightsquigarrow -m_k-1$.

To proceed we will express the higher $J$-modes (3.41) in terms of derivatives. Noting that $(L_{-1})^n J_{-1}|0\rangle = n!J_{-n-1}|0\rangle$ we see that the field corresponding to the state $J_{-n-1}|0\rangle$ is $\frac{1}{n!}\partial^n J(z)$. Using this, we finally find

$$\widetilde{W}^{(s)}(z) = \sum_{n=1}^{s/2}\sum_{1\leq i_1<\ldots<i_n\leq r}\sum_{\substack{m_1,\dots,m_n\geq 0\\ m_1+\cdots+m_n=s-2n}} \frac{(J^{\varepsilon_{i_1}}\partial^{m_1}J^{\varepsilon_{i_1}}\cdots J^{\varepsilon_{i_n}}\partial^{m_n}J^{\varepsilon_{i_n}})(z)}{m_1!\cdots m_n!}\,. \qquad (3.42)$$

With the help of (2.60), we can now write down the $r$ loop equations resulting from the Casimir fields $\{W^{(r)}(z), \widetilde{W}^{(2)}(z), \widetilde{W}^{(4)}(z), \dots, \widetilde{W}^{(2r-2)}(z)\}$. For $W^{(r)}(z)$ one finds

$$Z^{\rm reg}_{\rm mm}\left[y_{\varepsilon_1}(x)y_{\varepsilon_2}(x)\cdots y_{\varepsilon_r}(x) + P_r(x)\right] = 0\,, \qquad (3.43)$$

where $P_r(x)$ is a polynomial of degree $\leq rD$, while for $\widetilde{W}^{(s)}(z)$ one gets

$$Z^{\rm reg}_{\rm mm}\left[\sum_{n=1}^{s/2}N^{2n-s}\sum_{1\leq i_1<\ldots<i_n\leq r}\sum_{\substack{m_1,\dots,m_n\geq 0\\ m_1+\cdots+m_n=s-2n}}\prod_{k=1}^{n}\frac{y_{\varepsilon_{i_k}}(x)\,\partial^{m_k}y_{\varepsilon_{i_k}}(x)}{m_k!} + \tilde{P}_s(x)\right] = 0\,, \qquad (3.44)$$

where $\tilde{P}_s(x)$ is a polynomial of degree $\leq sD$. The $y_{\varepsilon_k}(x)$ are related to the resolvents via (2.53). Explicitly, defining also $\omega_0(x) \equiv 0$,

$$\begin{aligned} y_{\varepsilon_k}(x) &= \omega_k(x) - \omega_{k-1}(x) + \delta_{k,r-1}\omega_r(x) + t_k(x)\,,\\ t_k(x) &= -\frac{1}{Ng_s}(\varepsilon_k, W'(x)) = -\frac{1}{Ng_s}\left(\sum_{i=k}^{r-2}W'_i(x) + \frac{1}{2}\left(W'_r(x) + (-1)^{\delta_{k,r}}W'_{r-1}(x)\right)\right). \qquad (3.45)\end{aligned}$$

Again this relation can be inverted to give the resolvents $\omega_i$ in terms of the $y_{\varepsilon_i}$,

$$\omega_i(x) = \frac{1}{2}\left(2-\delta_{i,r-1}-\delta_{i,r}\right)\sum_{k=1}^{i}\left(y_{\varepsilon_k}(x) - t_k(x)\right) - \frac{1}{2}\,\delta_{i,r-1}\left(y_{\varepsilon_r}(x) - t_k(x)\right). \qquad (3.46)$$

Note that in contrast to $A_r$, the $D_r$-loop equations (3.44) in general contain derivatives of the variables $y_{\varepsilon_k}(x)$. These derivatives appear because the corresponding Casimir field (3.41) contains modes $J_{-m}$ with $m > 1$. Recall that this was not the case in the generating function (3.6) for the $A_r$ Casimir fields.

However, three of the Casimir fields can be expressed through the modes $J^{\varepsilon_k}_{-1}$ alone. The first is, by definition, $W^{(r)}(z)$ in (3.35). The second and the third are the stress tensor $T(z)$ and a spin 4 field $\mathrm{U}(z)$, defined as

$$\begin{aligned} T(0)|0\rangle &= \frac{1}{2}\sum_{i=1}^{r} J^{\varepsilon_i}_{-1}J^{\varepsilon_i}_{-1}|0\rangle = \frac{1}{2}\widetilde{W}^{(2)}(0)|0\rangle\,, \\ \mathrm{U}(0)|0\rangle &= \frac{1}{2}\sum_{1\le i<j\le r} J^{\varepsilon_i}_{-1}J^{\varepsilon_i}_{-1}J^{\varepsilon_j}_{-1}J^{\varepsilon_j}_{-1}|0\rangle - \frac{1}{4}\sum_{i=1}^{r} J^{\varepsilon_i}_{-1}J^{\varepsilon_i}_{-1}J^{\varepsilon_i}_{-1}J^{\varepsilon_i}_{-1}|0\rangle \\ &= \widetilde{W}^{(4)}(0)|0\rangle - L_{-2}L_{-2}|0\rangle\,. \end{aligned} \tag{3.47}$$

The last equality can be verified using

$$L_m = \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{i=1}^{r} {:}J^{\varepsilon_i}_k J^{\varepsilon_i}_{m-k}{:}\,. \tag{3.48}$$

Further, to see that $\mathrm{U}(z)$ is indeed a Casimir field one can use that $\widetilde{W}^{(4)}(z)$ is a Casimir field and that the screening charge zero modes commute with the Virasoro generators, $[V_{\alpha_i,0}, L_m] = 0$. This in turn follows from the commutation relation of the $L_m$ with the modes $K_n$ of any primary spin one field $K(z)$, which read $[L_m, K_n] = -nK_{n+m}$.

The Casimir fields $W^{(r)}(z)$, $T(z)$ and $\mathrm{U}(z)$ thus result in loop equations which do not involve derivatives. Note that this fits well with the observation that $D_3 = A_3$, as Lie algebras, and the three Casimir fields of $A_3$ with spins 2, 3 and 4 do not contain derivatives. For the $\widetilde{W}^{(s)}(z)$ with $s \ge 6$, linear combinations similar as in (3.47) do not seem to exist.

### 3.5 $D_r$-quiver model loop equations at large $N$

When taking the large $N$ limit of (3.44) only the coefficient of $N^0$ survives. For this coefficient we have $n = s/2$, so that it does not contain any derivatives. Abbreviating again $\tilde{y}_{\varepsilon_k}(x) = Z^{\mathrm{reg}}_{\mathrm{mm}}[y_{\varepsilon_k}(x)]/Z^{\mathrm{reg}}_{\mathrm{mm}}$ the large $N$ limit of the loop equations (3.43) and (3.44) reads

$$\tilde{y}_{\varepsilon_1}(x)\tilde{y}_{\varepsilon_2}(x)\cdots\tilde{y}_{\varepsilon_r}(x) = -P_r(x) \tag{3.49}$$

and

$$\sum_{1\le i_1<\ldots<i_{s/2}\le r} \tilde{y}_{\varepsilon_{i_1}}(x)^2\cdots\tilde{y}_{\varepsilon_{i_{s/2}}}(x)^2 = -\tilde{P}_s(x)\,, \tag{3.50}$$

where $s = 2, 4, \ldots, 2r-2$. As opposed to their finite $N$ form, the large $N$ loop equations can easily be written in terms of a generating function,

$$\prod_{k=1}^{r}(z + \tilde{y}_{\varepsilon_k}(x)^2) = -\sum_{k=0}^{r}\tilde{P}_{2r-2k}(x)z^k\,, \tag{3.51}$$

where we set $\tilde{P}_0(x) = -1$ and $\tilde{P}_{2r}(x) = -P_r(x)^2$. Note that (3.51) is weaker than the set of equations (3.49) and (3.50) because the coefficient of $z^0$ just gives the square of relation (3.49). From (3.51) we see that the $\tilde{y}_{\varepsilon_k}(x)$ are equal to $\pm i$ times the $r$ zeros of the polynomial in $z$ given by $\sum_{k=0}^{r}\tilde{P}_{2r-2k}(x)z^k = 0$ with the constraint that the signs of the $\tilde{y}_{\varepsilon_k}(x)$ have to be chosen such that equation (3.49) holds.

Similar as in the $A_r$ case, the large $N$ loop equations of the $D_r$-quiver matrix model is the deformation of the reduction of a Calabi-Yau three-fold to a nodal curve. The non-compact threefold is a fibration of a $D_r$-singularity over the complex $x$-plane with defining equation [36, 29, 2],

$$u^2 + v^2 z + \frac{1}{z}\left(\prod_{k=1}^{r}(z + t_k(x)^2) - \prod_{k=1}^{r} t_k(x)^2\right) + v\prod_{k=1}^{r} t_k(x) = 0\,, \qquad (3.52)$$

where $t_k(x)$ are given by (3.45). The deformation of nodes is achieved by by replacing $t_k(x)$ with $\tilde{y}_{\epsilon_k}(x)$ and turning on the $DR_+$ parameters. Note that due to the $Z_2$ symmetry of (3.51) pairs of douple points are resolved by this. The reduction of the holomorphic $(3,0)$ on the curve gives meromorphic forms as in (3.27).

## 4. Relation to Casimir algebras

The aim of this section is to relate the calculations in sections 2 and 3 to a special form of $\mathcal{W}$-algebras, the so-called Casimir algebras. We will start by some general comments on $\mathcal{W}$-algebras.

### 4.1 Some generalities on $\mathcal{W}$-algebras

By a chiral algebra one denotes a subsector of a full conformal field theory which consists only of holomorphic fields, i.e. of fields $\phi(z,\bar{z})$ which obey $\partial/\partial\bar{z}\,\phi(z,\bar{z}) = 0$. The chiral algebra need not contain all such fields, the only requirements are that it closes under the OPE and that it contains the conformal stress tensor $T(z)$.

Chiral algebras can be studied as objects on their own right, without reference to the original full CFT. This leads to the notion of a conformal vertex algebra, see e.g. [38] for a mathematical exposition of the subject. In fact, a standard approach to study CFTs is to start with a chiral algebra and to use its representation theory to construct the full CFT. This has lead for example to the Virasoro minimal models and the WZW-models, see e.g. [28].

For the Virasoro minimal models, the chiral algebra is just the Virasoro algebra, generated only by the stress tensor $T(z)$ itself. For WZW models the chiral algebra is a current algebra, that is, it is generated by fields of spin one. An obvious generalisation is to study chiral algebras that are still generated by a *finite* number of fields, but where the fields can have any integer spin. These are the $\mathcal{W}$-algebras; an extensive review can be found in [16]. $\mathcal{W}$-algebras made their appearance in [39], where a chiral algebra generated by $T(z)$ and an additional field of spin three was investigated.

A qualitative distinction between the Virasoro or current algebras and general $\mathcal{W}$-algebras is that the OPE of two generating fields is no longer a linear expression in the generators. On the level of modes this implies that the algebra of modes of the generating fields is not a Lie algebra. The commutator of two modes can contain (infinite sums of) products of modes.

$\mathcal{W}$-algebras appear in the context of matrix models and integrable hierarchies in the form of $\mathcal{W}$-constraints. In this case one has a set of differential operators $W_n$ which annihilate the $\tau$-function of the hierarchy and which do not form a Lie-algebra. Instead their commutators can give non-linear combinations of the $W_n$, and the resulting structure is that of the mode algebra of a conformal $\mathcal{W}$-algebra [13, 14].

A link between the $\mathcal{W}$-constraints in the matrix model context and the $\mathcal{W}$-algebras of conformal field theory is provided by the free boson representation of the matrix model, see [10]–[12]. In this paper we worked in the free boson representation right from the start. Another link, this time more generally between the Hirota bilinear equations for $\tau$-functions and $\mathcal{W}$-algebras, is given by the orbit construction, see e.g. [40] and references therein.

### 4.2 Casimir algebras and WZW-models

The $\mathcal{W}$-algebras we are interested in for the purposes of this paper are of a special type, called Casimir algebras. These are defined as follows, see [15] as well as [16] and references therein. Consider a WZW-model $\hat{\mathsf{g}}_k$ at level $k$ for a Lie algebra $\mathsf{g}$. The modes $T^i_m$ of the spin one currents $T^i(z)$ that generate the chiral algebra span an affine Lie algebra at level $k$, which we will also denote by $\hat{\mathsf{g}}_k$,

$$[T^i_m\,, T^j_n] = \sum_{\ell=1}^{\dim \mathsf{g}} f^{ij}{}_\ell\, T^\ell_{m+n} + km\, K(T^i, T^j)\, \delta_{m+n,0}\,, \tag{4.1}$$

where $T^i$ denote the generators of the underlying finite Lie algebra $\mathsf{g}$, the $f^{ij}{}_\ell$ are the structure constants of $\mathsf{g}$ and $K(\cdot,\cdot)$ is the Killing form on $\mathsf{g}$. Note that $\mathsf{g}$ is canonically embedded into $\hat{\mathsf{g}}_k$ via $T^i \mapsto T^i_0$. Denote this embedding by $\iota$. Similar to the coset construction of conformal field theory we can consider all elements in $\hat{\mathsf{g}}_k$ that commute with all elements of $\iota(\mathsf{g})$.

**Definition 3.** *The* Casimir algebra $\mathcal{W}[\hat{\mathsf{g}}_k/\mathsf{g}]$ *is defined as the (vertex-) subalgebra of the chiral algebra* $\hat{\mathsf{g}}_k$ *obtained by taking all fields* $\phi(z)$ *of* $\hat{\mathsf{g}}_k$ *that obey* $[T^i_0, \phi(z)] = 0$ *for all generators* $T^i$ *of* $\mathsf{g}$.

Via the state-field correspondence we can give an equivalent characterisation of $\mathcal{W}[\hat{\mathsf{g}}_k/\mathsf{g}]$ in terms of states in the vacuum module of $\hat{\mathsf{g}}_k$,

$$[T^i_0, \phi(z)] = 0 \quad \Leftrightarrow \quad T^i_0\phi(0)|0\rangle = 0\,. \tag{4.2}$$

The chiral algebra $\hat{\mathsf{g}}_k$ is spanned as a vector space by fields of the form

$$W^{i_1\ldots i_n}_{m_1\ldots m_n}(z) = (\partial^{m_1}T^{i_1}\cdots\partial^{m_n}T^{i_n})(z)\,. \tag{4.3}$$

One can verify (see [16]) that, for generic level $k$, the linear combination

$$W_{m_1\ldots m_n}(z) = \sum_{i_1,\ldots,i_n} d_{i_1\ldots i_n} W^{i_1\ldots i_n}_{m_1\ldots m_n}(z) \tag{4.4}$$

is in $\mathcal{W}[\hat{\mathsf{g}}_k/\mathsf{g}]$ if and only if $\sum_{i_1,\dots,i_n} d_{i_1\dots i_n} T^{i_1}\cdots T^{i_n}$ is in the centre of the universal enveloping algebra of $\mathsf{g}$, i.e. if it is a Casimir element of $\mathsf{U}(\mathsf{g})$. This is the reason for the terminology 'Casimir algebra'. For certain specific values of $k$, however, one looses the 'only if' in the above relation (just take the free boson stress tensor in the free boson realisation of $\hat{\mathsf{g}}_1$ as an example).

### 4.3 Casimir fields and Casimir algebras

We would like to find the relation between the notion of a 'Casimir field' used in sections 2 and 3 and the Casimir algebras introduced above.

Choose a Cartan-Weyl basis for $\mathsf{g}$. In the free field realisation of $\hat{\mathsf{g}}_1$, the free boson currents $J^{(a)}(z)$ provide the $\hat{\mathsf{g}}_1$-fields $H^i(z)$, where $H^i$ denotes an element of the Cartan subalgebra of $\mathsf{g}$. The $\hat{\mathsf{g}}_1$-fields $E^{\alpha_i}(z)$, for the ladder operators $E^{\alpha_i} \in \mathsf{g}$ and $\alpha_i$ a simple root, are given by the free boson vertex operators $V_{\alpha_i}(z)$, up to a cocycle factor, which we do not spell out explicitly, see e.g. [28, section 15.6.3]. We would like to establish

$$\phi(z) \text{ is a Casimir field} \qquad \Leftrightarrow \qquad \phi(z) \in \mathcal{W}[\hat{\mathsf{g}}_1/\mathsf{g}]\,. \tag{4.5}$$

It is enough to check the defining condition in (4.2) for the modes $E_0^{\alpha_i}$ and $H_0^i$ for $i = 1,\dots,r$. If $\phi(z)$ is a Casimir field, then $J_0^{(a)}\phi(0)|0\rangle = 0$ since by construction the state $\phi(0)|0\rangle$ is in the vacuum module of the $\mathrm{U}(1)^r$-algebra and thus has $\mathrm{U}(1)^r$-charge zero. Furthermore, by definition 1 in section 2.5 we have $V_{\alpha_i,0}\phi(0)|0\rangle = 0$, so that indeed $\phi(z) \in \mathcal{W}[\hat{\mathsf{g}}_1/\mathsf{g}]$.

In the converse direction we have to be more careful, because in the definition we used, Casimir fields are build *only* from the $\hat{\mathsf{g}}_1$-currents $H^i(z)$, corresponding to the Cartan subalgebra of $\mathsf{g}$, while the fields in $\mathcal{W}[\hat{\mathsf{g}}_1/\mathsf{g}]$ are constructed from *all* generators $T^i(z)$ in $\hat{\mathsf{g}}_1$. Suppose $\phi(z) \in \mathcal{W}[\hat{\mathsf{g}}_1/\mathsf{g}]$. Then because $H_0^i\phi(0)|0\rangle = 0$ we know that the state $\phi(0)|0\rangle$ is indeed in the vacuum module of the $\mathrm{U}(1)^r$-algebra and can thus be expressed in terms of the modes $J_{-m}^{(a)}$ alone. The remaining conditions $E_0^{\alpha_i}\phi(0)|0\rangle = 0$ then imply that $\phi(z)$ is a Casimir field.

In the $\mathsf{g} = A_r$ and $\mathsf{g} = D_r$ examples discussed in section 3 we have constructed $r$ fields of $\mathcal{W}[\hat{\mathsf{g}}_1/\mathsf{g}]$. One can now wonder if by repeatedly taking normal ordered products, these fields (and their derivatives) generate all of $\mathcal{W}[\hat{\mathsf{g}}_1/\mathsf{g}]$. There are arguments (see [16] and references therein), using the Drinfeld-Sokolov reduction and the character technique, that this in indeed the case.

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## A. Integral form of $\mathcal{W}$-constraints

In this appendix we establish equation (2.41). Denote by $I(L)$ the integral in (2.41) before taking the limit. In the correlator $Z_{\text{cft}}^{\text{reg}(\varepsilon,L)}[W(z)]$ there are $N = N_1 + \cdots + N_r$ screening integrals, one along each of the contours $\gamma_{k,K}^{\varepsilon,L}$. Let us denote

$$Q_k(\gamma_{k,K}^{\varepsilon,L}) = \oint_{\gamma_{k,K}^{\varepsilon,L}} \frac{dx}{2\pi i} V_{\alpha_k}(x)\,. \tag{A.1}$$

The integration contour $\gamma_{\text{ev}}$ of $W(z)$ can be deformed to encircle each of the $\gamma_{k,K}^{\varepsilon,L}$. As in (2.38), carrying out the $z$ and the corresponding screening integration results in the insertion of a pair of fields $A(y_1){-}A(y_0)$ at the endpoints of the encircled contour $\gamma_{k,K}^{\varepsilon,L}$. The function $I(L)$ can thus be rewritten as follows,

$$\begin{aligned} I(L) &= \oint_{\gamma_{\text{ev}}} \frac{dz}{2\pi i}\,\frac{1}{z-x}\,\langle\vec{N}|e^{-\mathsf{H}} \prod_{j,J} Q_j(\gamma_{j,J}^{\varepsilon,L})\; W(z)|0\rangle \\ &= \sum_{k,K} \langle\vec{N}|e^{-\mathsf{H}} \prod_{j,J\neq k,K} Q_j(\gamma_{j,J}^{\varepsilon,L}) \left(A_k\left(y_1^{k,K}\right) - A_k\left(y_0^{k,K}\right)\right)|0\rangle\,. \end{aligned} \tag{A.2}$$

Here $y_1^{k,K} = \gamma_{k,K}^{\varepsilon,L}(L)$, $y_0^{k,K} = \gamma_{k,K}^{\varepsilon,L}(-L)$ and $A_k(y)$ denotes the field (2.39) with $V$ replaced by $V_{\alpha_k}$ and $f(y) = (y{-}x)^{-1}$. The individual field insertions $A_k(y)$ are a sum of terms of the form

$$(\text{const})\cdot \frac{1}{(y-x)^p}\;\partial^q\{V_{\alpha_k}W\}_r(y)\,. \tag{A.3}$$

The state $|\psi\rangle = \partial^q\{V_{\alpha_k}W\}_r(0)|0\rangle$ is an element in the $\mathrm{U}(1)^r$-highest weight representation $\mathcal{H}_{\alpha_k}$ of charge $\alpha_k$ and with highest weight vector $|\alpha_k\rangle$. The space $\mathcal{H}_{\alpha_k}$ is spanned by the states $J^{u_1}_{-m_1-1}\cdots J^{u_n}_{-m_n-1}|\alpha_k\rangle$, for $m_i \geq 0$ and $u_i \in \mathsf{h}^*$. Thus the field $\partial^q\{V_{\alpha_k}W\}_r(y)$ is itself a linear combination of fields of the form

$$\phi(y) = (\partial^{m_1}J^{u_1}\cdots\partial^{m_n}J^{u_n}V_{\alpha_k})(y)\,. \tag{A.4}$$

In order to find an estimate for the integral $I(L)$ we will first establish the formula

$$\langle\vec{N}|e^{-\mathsf{H}}V_{q_1}(x_1)\cdots V_{q_n}(x_n)\phi(y)|0\rangle = \frac{P(x_1,\ldots,x_n,y)}{Q(x_1,\ldots,x_n,y)}\langle\vec{N}|e^{-\mathsf{H}}V_{q_1}(x_1)\cdots V_{q_n}(x_n)V_{\alpha_k}(y)|0\rangle\,, \tag{A.5}$$

where $\sum_i q_i + \alpha_k = \vec{N}$, and $P$, $Q$ are polynomials, whose detailed form will not matter. Take a field $\tilde{\phi}(y)$ which is also of the form (A.4) and consider the function $g(y)$ defined as

$$g(y) = \langle\vec{N}|e^{-\mathsf{H}}\;V_{q_1}(x_1)\cdots V_{q_n}(x_n)\;(\partial^m J^u\tilde{\phi})(y)\,|0\rangle\,. \tag{A.6}$$

Using the definition of the normal ordered product we can write

$$\begin{aligned} g(y) &= \oint_{\gamma_y}\frac{dz}{2\pi i}\,\frac{1}{z-y}\langle\vec{N}|e^{-\mathsf{H}}\;V_{q_1}(x_1)\cdots V_{q_n}(x_n)\;\partial^m J^u(z)\;\tilde{\phi}(y)\,|0\rangle \\ &= \left(\oint_{\gamma_\infty}\frac{dz}{2\pi i} - \sum_{i=1}^n\oint_{\gamma_{x_i}}\frac{dz}{2\pi i}\right)\frac{m!}{(z-y)^{m+1}}\langle\vec{N}|e^{-\mathsf{H}}\;V_{q_1}(x_1)\cdots V_{q_n}(x_n)\;J^u(z)\;\tilde{\phi}(y)\,|0\rangle \\ &= \left(-\frac{1}{g_s}\left(u,\partial^{m+1}\mathsf{W}(y)\right) - \sum_{i=1}^n\frac{m!\;(q_i,u)}{(x_i-y)^{m+1}}\right)\langle\vec{N}|e^{-\mathsf{H}}\;V_{q_1}(x_1)\cdots V_{q_n}(x_n)\;\tilde{\phi}(y)\,|0\rangle\,. \end{aligned} \tag{A.7}$$

In the first step, the integration contour $\gamma_y$ has been deformed to encircle the points $x_i$ as well as the point $\infty$. Also, by partial integration the $m$ derivatives $\partial/\partial z$ have been shifted from $J^u(z)$ to the factor $(z-y)^{-1}$. The last step uses the OPE (2.23) as well as the fact that due to (2.43) the asymptotic behaviour of the correlator in the second line of (A.7) is given by

$$\begin{aligned}&\langle\vec{N}|e^{-\mathsf{H}}\ V_{q_1}(x_1)\cdots V_{q_n}(x_n)\ J^u(z)\ \tilde{\phi}(y)\,|0\rangle = \\ &\quad = -\frac{1}{g_s}(u,\mathsf{W}'(z))\langle\vec{N}|e^{-\mathsf{H}}\ V_{q_1}(x_1)\cdots V_{q_n}(x_n)\tilde{\phi}(y)\,|0\rangle + O(z^{-1})\,. \end{aligned}\tag{A.8}$$

Furthermore, it was used that for the contour integration around infinity we have, for some polynomial $p(z)$,

$$\oint_{\gamma_\infty}\frac{dz}{2\pi i}\left[\big(p(z)+O(z^{-1})\big)\,(-1)^m\frac{\partial^m}{\partial z^m}\frac{1}{z-y}\right] = \partial^m p(y)\,. \tag{A.9}$$

Applying this procedure recursively allows us to strip a field $\tilde{\phi}(y)$ of the form (A.4) of all its components $\partial^{m_i}J^{u_i}$, establishing equation (A.5).

With the help of (A.5) we can now find an estimate for a correlator involving screening integrals. Consider the equalities

$$\begin{aligned}&\langle\vec{N}|e^{-\mathsf{H}}\prod_{j,J\neq k,K}Q_j(\gamma^{\varepsilon,L}_{j,J})\phi(y)\,|0\rangle = \\ &= \left(\prod_{j,J\neq k,K}\oint_{\gamma_{j,J}}\frac{d\lambda_{j,J}}{2\pi i}\right)\frac{P(\lambda_{1,1},\ldots,\lambda_{r,N_r},y)}{Q(\lambda_{1,1},\ldots,\lambda_{r,N_r},y)}\langle\vec{N}|e^{-\mathsf{H}}\prod_{r,R\neq k,K}V_{\alpha_r}(\lambda_{r,R})\ \ V_{\alpha_k}(y)\,|0\rangle \\ &= \left[\left(\prod_{j,J\neq k,K}\oint_{\gamma_{j,J}}\frac{d\lambda_{j,J}}{2\pi i}\right)\frac{\tilde{P}(\lambda_{1,1},\ldots,\lambda_{r,N_r},y)}{\tilde{Q}(\lambda_{1,1},\ldots,\lambda_{r,N_r},y)}e^{-\frac{1}{g_s}\sum_{r,R\neq k,K}W_r(\lambda_{r,R})}\right]e^{-\frac{1}{g_s}W_k(y)}\,. \end{aligned}\tag{A.10}$$

Here $P, Q, \tilde{P}, \tilde{Q}$ are polynomials whose arguments consist of $y$ as well as of all $\lambda$'s except for $\lambda_{k,K}$. The point $y$ depends on $L$ and is taken to have either of two values

$$y = y(L) = \gamma^{\varepsilon,L}_{k,K}(\pm L)\,. \tag{A.11}$$

In the first step in (A.10) the screening integrals have been written out explicitly and (A.5) was substituted. In the second step the explicit form of the correlator, obtained from (2.21), (2.23) and (2.25), has been inserted and the rational part of the expression has been absorbed by redefining the polynomials $P$ and $Q$ appropriately.

Denote by $F(L)$ the multiple integral inside the square brackets in the last line of equation (A.10). For finite $L$ this integral is finite, since the only singularities of the integrand could occur at coinciding integration variables $\lambda_{i,I} = \lambda_{j,J}$ which is impossible by construction of the regularised contours $\gamma^{\varepsilon,L}_{i,I}$. Furthermore, because of the asymptotic behaviour of the contour $\gamma(x)$ imposed at the end of section 2.4, the integrand of $F(L)$ receives an exponential damping as its arguments approach infinity. We can conclude that the $L\to\infty$ limit of $F(L)$ is well defined,

$$\lim_{L\to\infty}F(L) = C\,, \tag{A.12}$$

for some constant $C$. For the correlator in the first line of (A.10) this implies that

$$\lim_{L\to\infty} \langle \vec{N}| e^{-\mathsf{H}} \prod_{j,J\neq k,K} Q_j(\gamma_{j,J}^{\varepsilon,L}) \quad \phi(y)\,|0\rangle = \lim_{L\to\infty} F(L) e^{-\frac{1}{g_s}W_k(y(L))} = 0\,. \tag{A.13}$$

Since from (A.2) we know that $I(L)$ can be written as a linear combination of terms of the form (A.10), the above equation proves that indeed $\lim_{L\to\infty} I(L) = 0$, as claimed.

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